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Linearized IIA Supergravity Equations of Motions on $\mathrm{AdS}_7 \times S^3$ Background

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Ai miei genitori,
la mia vita é il loro piú grande dono.

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1 Introduction

Since this work has its roots in one of the known string theories, namely type IIA Superstring theory, let's start by taking a look at the general features of such theories.

String theory is one of best the candidates for a quantum theory of gravity, it unifies all known forces and particles (while suggesting the existence of many more) as different vibrational modes of an elementary string. These elementary strings can be open or closed depending on the number of endpoints they have: open strings have two endpoints while closed strings have none. They can interact with each other, open strings can join to form closed strings and closed strings can split into open strings.

One of the interesting features of such theories is that they include general relativity at long distances, this means that gravity is actually embedded in a quantum theory and it is possible to take it into account also at short distances with a string theory description. This would solve the long standing problem of fitting gravity into a quantum field theory framework.

Another interesting feature of the theory is that the dimensionality of space-time does not need to be fixed before working out the details of the model but it emerges from a calculation. Contrary to our expectations, it turns out that the theory needs ten space-time dimensions to be consistent; this gives rise to the need for a special mechanism to bring down the number of dimensions to the observable four ones we are used to. One way to achieve this goal has been to "compactify" the extra spatial dimensions, this means that those dimensions are made so small we have no access to them through our experiments and so we do not notice their existence.

Also, Supersymmetry is a basic ingredients in string theories which have the characteristic to resemble reality. In fact, there exist bosonic string theories too, which lack fermions and have a negative mass particle (the tachyon) in the spectrum, that are not seen as a good description of our reality because of these deficiencies. Supersymmetry is actually required for the mathematical consistency of the theory and it is one of the main predictions that could be testable at accessible energies.

In this thesis we will mainly discuss the linearization procedure of the equations of motion of type IIA supergravity, the low-energy limit of type IIA superstring theory. In particular, our goal is to linearize those equations and express them in terms of the fluctuations of the fields on a $AdS_7 \times M_3$ background space-time. Once we know those equations, we could use them to determine the spectrum of masses of the fields of the theory by expanding the fluctuations in harmonics on M_3 and diagonalizing the linearized equations. Then, with the help of the AdS/CFT correspondence, we could relate the mass of the supergravity fields of AdS_7 with the index of conformal operators in CFT_6. In this way, we can learn more about CFT_6, which is the field theory describing the scalar fields living on the M5-brane of M-theory. Since those scalar fields come from the coordinates describing the position of the brane in the directions perpendicular to it, they provide us a way to learn about the membrane itself.

In order to carry on the linearization procedure succesfully, we need to introduce all the necessary concepts needed to understand the topic thoroughly. For this reason, the first part of the thesis is devoted to the introduction of those basic ideas which will be useful later, to get to the specific case we will be dealing with; while the second part focuses on the computations and the results obtained from the case of interest. The second section (2) is used to present the basic concepts of string theory. We discuss the case of the bosonic and super string in details; the action, the equations of motion and the quantization are some of the topics we investigate.

Section 3 is concerned with the schematic treatment of supersymmetry and supergravity. This is useful because supersymmetry is a key concept in superstring theories. Moreover, it is important to introduce the basics of supergravity because the case we'll be treating is exactly one kind of supergravity theory. In particular, we will study type IIA supergravity in ten dimensions on a specific background.

The next section, number 4, is devoted to Kaluza-Klein supergravity theories. In those kind of theories we employ the Kaluza-Klein idea of unifying the fields of a theory by adding compact dimensions that are not usually accesible experimentally, to interpret supergravity theories in dimensions greater than four. We start from a supergravity theory and apply certain compactification schemes (configurations of the background space-time) which make the expression of the fields take a peculiar form. For example, in the case at hand we take the background ten dimensional space-time to be split into a seven dimensional anti-de Sitter space-time and a three dimensional compact space. Given this configuration for the background space, the expressions of the fields of the theory were found by solving the equations of motion. These values were then taken to be the ground-state values of the fields.

Also, the last section of the first part, section 5, goes into more details about the theory we are studying and the mathematical techniques that were used to determine the properties of interest. The relation between supergravity theories and their consecuences on the geometry of space-time is briefly discussed. Furthermore, we give some highlights of complex geometry and generalized complex geometry, which are needed to rewrite in a more simple form the supersymmetry transformations of the theory and to make them easier to solve. Lastly, the main results about type IIA supergravity obtained by using those techniques are listed.

The second part of the thesis, which consists of section 6, deals with the computations needed to linearize the equations of motion of the theory we study: type IIA supergravity on an $\mathrm{AdS}_7 \times M_3$ background. In section 6, we have carried out the computation for the bosonic fields only and in particular we focused on the equations of motion for the forms, the dilaton equation and the Einstein equation. We start with the equations of motion of type IIA supergravity and we linearize them by expanding the fields around their ground-state value to first order in the perturbations. Once we obtain the linearized equations, we substitute the known expression of the background fields and we rewrite the equations making the fact that we have a specific background space explicit. For instance, the metric will have a certain form and the expressions of the fields will have to agree with the symmetries of the background space. In this way we get the linearized equations for our case.

2 String theory

2.1 Point particles and strings in curved space-time

2.1.1 General relativity in M dimensions

We consider the action of general relativity together with scalar fields and a cosmological constant:

$$S = \int d^M x \sqrt{-g} \left(R - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^j M_{ij} - \Lambda \right), \tag{2.1}$$

where the expression $\partial_\mu \varphi^i \partial_\nu \varphi^j M_{ij}$ denotes the presence of several scalar fields.

The action has the following symmetries:

- diffeomorphism invariance;
- global symmetry $\varphi^i \to \varphi^i + a^i$;
- $\varphi^i \to \Lambda^i_l \varphi^l$ so that $\Lambda^* M \Lambda = M$.

The equations of motion can be obtained by varying the action. For $g_{\mu\nu}$ we get the Einstein equation:

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0, \tag{2.2}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0. \tag{2.3}$$

$M = 1$ dimension

If we specialize the general case of general relativity in M-dimension to the case of $M = 1$ we have:

$$x^0 \to \tau \ , \qquad R \to 0 \ ,$$

$$M_{ij} \to \eta_{\mu\nu} \ , \qquad \varphi^i \to X^\mu(\tau) \ , \qquad g \to -e^2 \ ;$$

where the metric only has one component.

The action now becomes:

$$S[e, X] = \int d\tau e \left[\frac{1}{2} \left(\frac{1}{e^2} \right) \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau} \eta_{\mu\nu} - \Lambda \right] = \frac{1}{2} \int d\tau \left[\frac{1}{e} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau} \eta_{\mu\nu} - 2e\Lambda \right]. \quad (2.4)$$

The equation of motion for e is:

$$\frac{\delta S}{\delta e} = 0, \quad (2.5)$$

$$-\frac{1}{e^2} \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - 2\Lambda = 0 \ \to \ e^2 = -\frac{\dot{X}^2}{2\Lambda}; \quad (2.6)$$

where we have defined: $\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau}$ and $\dot{X}^2 = \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}$.

Substituting the expression for e which solves the equation of motion we get:

$$S[X] = \frac{1}{2} \int d\tau \left[\sqrt{-2\Lambda \dot{X}^2} + \sqrt{-2\Lambda \dot{X}^2} \right] = \int d\tau \sqrt{-2\Lambda \dot{X}^2}. \quad (2.7)$$

Non-relativistic limit If we take the non-relativistic limit of the above expression, which means we take:

$$\dot{X}^2 = -\left(\tfrac{dt}{d\tau}\right)^2 + \left(\tfrac{d\vec{x}}{d\tau}\right)^2 \text{ and } \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} ,$$

the action can be rewritten as:

$$S = -\sqrt{2\Lambda} \int dt \sqrt{1 - \left(\frac{d\vec{x}}{dt} \right)^2} \approx -\sqrt{2\Lambda} \int dt \left(1 - \frac{1}{2} \left(\frac{d\vec{x}}{dt} \right)^2 \right); \quad (2.8)$$

which is the action for a point particle with mass: $m \equiv \sqrt{2\Lambda}$.

So, going back to the general case, we can rewrite the action in (2.7) :

$$S[X] = -m \int d\tau \sqrt{-\dot{X}^2}. \tag{2.9}$$

Now if the space-time is curved we substitute the flat metric $\eta_{\mu\nu}$ with the general metric $G_{\mu\nu}$ and we may write the action as:

$$S[X] = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}^\nu G_{\mu\nu}}. \tag{2.10}$$

As an aside, it is important to note that if we use the diffeomorphism invariance $\tau' = \tau'(\tau)$, we can impose the gauge fixing $e = 1$ and looking at the equation of motion for e we get the constraint: $P^2 + m^2 = 0$

$M = 2$ dimensions

$$\left(x^0, x^1\right) \to (\tau, \sigma)\,, \qquad M_{ij} \to \eta_{\mu\nu}\ ,$$

$$\varphi^i \to X^\mu(\tau, \sigma)\,, \qquad g_{mn} \to h_{mn}\ .$$

The most general expression we can write in this case for the Riemann tensor is:

$$R_{mnpq} = \alpha\left(h_{mp}h_{nq} - h_{mq}h_{np}\right)\ .$$

From this expression we can compute the form of the Ricci tensor R_{nq} and the curvature scalar R:

$$R_{nq} = h^{mp}R_{mnpq} = \alpha\left(2h_{nq} - h_{nq}\right) = \alpha h_{nq}\ ,$$

$$R = h^{nq}R_{nq} = 2\alpha\ .$$

So, the Einstein tensor $G_{nq} = R_{nq} - \frac{1}{2}h_{nq}R = 0$ vanishes.

The Einstein equation becomes

$$\Lambda h_{nq} = T_{nq}, \tag{2.11}$$

where the energy-momentum tensor T_{nq} is:

$$T_{nq} = \eta_{\mu\nu}\left(\partial_n X^\mu \partial_q X^\nu - \frac{1}{2} h_{nq} h^{mn} \partial_m X^\mu \partial_n X^\nu\right). \tag{2.12}$$

Taking the trace of the energy-momentum tensor we find:

$$T_n^n = h^{nq} T_{nq} = 0. \tag{2.13}$$

This means that if we take the trace of the Einstein equation in (2.11) we discover that:

$$h^{nq} \Lambda h_{nq} = 0 \ \rightarrow \ \Lambda = 0. \tag{2.14}$$

The cosmological constant must be null in this case.

The action will then be:

$$S = \int d\tau d\sigma \sqrt{-h}\left(R - \frac{1}{2} h^{mn} \partial_m X^\mu \partial_n X^\nu \eta_{\mu\nu}\right). \tag{2.15}$$

We can define: $\chi \equiv \int d\tau d\sigma \sqrt{-h} R$,

it is a number so it does not contribute to the action. More specifically, it is called the Euler-number and it is related to the genus of the surface spanned by (τ, σ).

The final form of the action is:

$$S = -\int d\tau d\sigma \sqrt{-h}\left(\frac{1}{2} h^{mn} \partial_m X^\mu \partial_n X^\nu \eta_{\mu\nu}\right), \tag{2.16}$$

and it is called the Polyakov action.

2.2 Bosonic string

2.2.1 Polyakov and Nambu-Goto action

Let's study here the properties of the Polyakov action:

$$S = -\frac{1}{2}\int d^2\sigma \sqrt{-h}\left(h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right) \ . \tag{2.17}$$

The equation of motion for $h_{\alpha\beta}$ is:

$$\partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X = 0 \; ; \tag{2.18}$$

which can be manipulated a bit in order to eliminate h from the action. For example, we rewrite it as:

$$\partial_\alpha X \cdot \partial_\beta X = \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X$$

taking the determinant we get:

$$-det\left(\partial_\alpha X \cdot \partial_\beta X\right) = -det\left(h_{\alpha\beta}\right) \left(\frac{1}{2} h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X\right)^2 \; ;$$

where $h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X$ is a scalar, so when we take the determinant of the 2×2 matrix $h_{\alpha\beta}$ this factor will appear twice. We define: $G_{\alpha\beta} \equiv (\partial_\alpha X \cdot \partial_\beta X)$.

Now, taking the square root:

$$\sqrt{-det\left(G_{\alpha\beta}\right)} = \sqrt{-h} \left(\frac{1}{2} h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X\right) \; . \tag{2.19}$$

Substituting this expression into the action we obtain what is called the Nambu-Goto action:

$$S\left[X\right] = -T \int d^2\sigma \sqrt{-det\left(G_{\alpha\beta}\right)} \; , \tag{2.20}$$

where $T = \frac{1}{2\pi\alpha'}$ is the string tension.

Symmetries

We list here the symmetry properties of S:

- diffeomorphism invariance (coordinate reparametrization): $(\tau, \sigma) \to (\tau', \sigma') = (\tau'(\tau, \sigma), \sigma'(\tau, \sigma))$;
- Weyl invariance: $h_{\alpha\beta} \to e^{\phi(\sigma,\tau)} h_{\alpha\beta}$, $\delta X^\mu = 0$;
- global symmetry: $\delta X^\mu(\tau, \sigma) = a^\mu_\nu X^\nu + b^\mu$, $\delta h_{\alpha\beta} = 0$.

We can use these symmetries to fix the metric to a Minkowski metric:

$$h_{\alpha\beta} = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} .$$

If we choose this gauge fixing for $h_{\alpha\beta}$ we can rewrite the Polyakov action as:

$$S[X] = -\frac{T}{2}\int d^2\sigma \left(\dot{X}^2 - X'^2\right) , \tag{2.21}$$

where $\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau}$, $X'^\mu \equiv \frac{\partial X^\mu}{\partial \sigma}$.

2.2.2 Equation of motion

By setting the variation of the action to zero we look for the field equations:

$$\delta S[X] = -\frac{T}{2}\int d^2\sigma \left(2\dot{X}\delta\dot{X} - 2X'\delta X'\right) = -T\int d^2\sigma \left[-\ddot{X}\delta X + \partial_\tau\left(\dot{X}\delta\dot{X}\right) + X''\delta X - \partial_\sigma\left(X'\delta X'\right)\right] =$$

$$= T\int d^2\sigma \left[\left(\ddot{X} - X''\right)\delta X - \partial_\tau\left(\dot{X}\delta X\right) + \partial_\sigma\left(X'\delta X\right)\right] = 0 , \tag{2.22}$$

where $-\partial_\tau\left(\dot{X}\delta\dot{X}\right) + \partial_\sigma\left(X'\delta X'\right)$ are boundary terms. The field equations are:

$$\left(\ddot{X}^\mu - X''^\mu\right) = 0 . \tag{2.23}$$

Or, written differently

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu(\tau,\sigma) = 0 . \tag{2.24}$$

From the gauge fixing we have chosen, we get some constraints on the equation of motion:

$$T_{\alpha\beta} = 0 , \tag{2.25}$$

$$T_{00} = T_{11} = 0 \quad T_{10} = T_{01} = 0 ,$$

$$\begin{cases} X'\cdot\dot{X} = 0 \\ \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0 . \end{cases} \tag{2.26}$$

2.2.3 Boundary conditions

We have two boundary terms two deal with:

1. $\int d\sigma \int_{-\infty}^{\infty} d\tau \partial_\tau \left(\dot{X}\delta X\right) \to 0$, by assumption we take it to vanish at infinity;
2. $\int d\tau \int_0^\pi d\sigma \partial_\sigma \left(X'\delta X\right) = \int d\tau \left[X'\delta X|_{\sigma=\pi} - X'\delta X|_{\sigma=0}\right] = 0$.

Let's see what are the possible conditions we can impose on the two terms.

For the case of **open strings**, whose endpoints can end at different space-time points, we can have:

- Neumann boundary condition: $X'^\mu(\tau,\pi) = X'^\mu(\tau,0) = 0$; the component of the momentum normal to the boundary of the world-sheet vanishes;
- Dirichlet boundary condition: $\begin{cases} X^\mu(\tau,\pi) = x_\pi \\ X^\mu(\tau,0) = x_0 \end{cases}$, where x_0 and x_π are constants; the positions of the two string ends are fixed so that $\delta X^\mu = 0$.

For **closed strings** the mapping is periodic $\left(X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi)\right)$, so the boundary term in the integral vanishes automatically:

- Periodic: $X^\mu(\tau,\pi) = X^\mu(\tau,0)$.

2.2.4 Light-cone coordinates

We present here the definitions of light cone coordinates because they are widely used in rewriting several equations.

We define world-sheet light cone coordinates:

$$\sigma^\pm \equiv \tau \pm \sigma \ . \tag{2.27}$$

And then we get:

$$\partial_\pm = \frac{1}{2}\left(\partial_\tau \pm \partial_\sigma\right) \ . \tag{2.28}$$

The two-dimensional flat metric becomes

$$\begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{2.29}$$

In these coordinates the wave equation for X^μ is now

$$\partial_+\partial_- X^\mu = 0 \ . \tag{2.30}$$

The most general solution can be written as $X^\mu(\sigma^+, \sigma^-) = X_R^\mu(\sigma) + X_L^\mu(\sigma^+)$.

From this expression we discover that $\partial_- X^\mu = \partial_- X_R^\mu$ and $\partial_+ X^\mu = \partial_+ X_R^\mu$.

These relations, combined with closed string boundary condition $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \pi)$, tell us that both X_L^μ and X_R^μ must be periodic, not just their sum X^μ.

Since they are periodic, we can expand them in Fourier modes:

$$\partial_- X_R^\mu = l_s \sum_{m=-n}^{+\infty} \alpha_m^\mu e^{-2mi\sigma_-} \ , \tag{2.31}$$

$$\partial_+ X_L^\mu = l_s \sum_{m=-n}^{+\infty} \tilde{\alpha}_m^\mu e^{-2mi\sigma_+} \ . \tag{2.32}$$

Integrating we obtain:

$$X_R^\mu = x_R^\mu + l_s \alpha_0 \sigma_- + \frac{il_s}{2} \sum_{m\neq 0}^{+\infty} \frac{\alpha_m^\mu}{m} e^{-2mi\sigma_-} \ , \tag{2.33}$$

$$X_L^\mu = x_L^\mu + l_s \tilde{\alpha}_0 \sigma_+ + \frac{il_s}{2} \sum_{m\neq 0}^{+\infty} \frac{\tilde{\alpha}_m^\mu}{m} e^{-2mi\sigma_+} \ , \tag{2.34}$$

$$X^\mu = (x_R^\mu + x_L^\mu) + l_s \left(\alpha_0 \sigma_-\tilde{} + \alpha_0 \sigma_+\right) + \frac{il_s}{2} \sum_{m\neq 0}^{+\infty} \frac{1}{m} \left(\alpha_m^\mu e^{-2mi\sigma_-} + \tilde{\alpha}_m^\mu e^{-2mi\sigma_+}\right) \ . \tag{2.35}$$

We also impose that X^μ is real: $(\alpha_m^\mu)^* = \alpha_{-m}^\mu$.

We now rewrite the constraints (2.26) in light-cone coordinates. They give

$$(\partial_+ X_L^\mu)^2 = 0 \quad (\partial_- X_R^\mu)^2 = 0 \tag{2.36}$$

For closed strings, they are periodic in σ so we can expand them in Fourier modes:

$$\begin{cases} (\partial_+ X_L^\mu)^2 = 2l_s^2 \sum_{m=-\infty}^{+\infty} \tilde{L}_m e^{-2im\sigma_+} \\ (\partial_- X_R^\mu)^2 = 2l_s^2 \sum_{m=-\infty}^{+\infty} L_m e^{-2im\sigma_-} \ . \end{cases} \tag{2.37}$$

To find the expression for L_m and $\tilde{L}_m$, which are called Virasoro operators, we can compare the above expressions with the definition of X^μ. Eventually, one gets:

$$\begin{cases} \tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \tilde{\alpha}_n^\mu \tilde{\alpha}_{m-n}^\nu \\ L_n = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \alpha_n^\mu \alpha_{m-n}^\nu \ . \end{cases} \tag{2.38}$$

In terms of the Virasoro operators, the constraints become:

$$L_n = \tilde{L}_m = 0 \ . \tag{2.39}$$

If we consider the zero mode we have: $\tilde{L}_0 = L_0 = 0$.

The expression of the zero mode is:

$$L_0 = \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n\neq 0} \alpha_n^\mu \alpha_{-n}^\nu \ , \tag{2.40}$$

we can rewrite the sum as: $\frac{1}{2}\sum_{n=-\infty}^{+\infty} \alpha_n^\mu \alpha_{-n}^\nu = \sum_{n>0} \alpha_n^\mu \alpha_{-n}^\nu$.
We also have $\alpha_0^2 = \frac{P_0^2}{(2\pi l_s T)^2} = \frac{-M^2}{(2\pi l_s T)^2}$.
So, from the constraint $L_0 = 0$ we find the following identity:

$$\sum_{m>0} \alpha_m^\mu \alpha_{-m}^\nu = \frac{1}{2}\frac{M^2}{(2\pi l_s T)^2} \ .$$

And from the definition of L_0 we get:

$$4\sum_{m>0} \alpha_m^\mu \alpha_{-m}^\nu = \alpha' M^2 \ . \tag{2.41}$$

Now, we can write a formula for the mass of the closed string:

$$\alpha' M_{closed}^2 = 4\sum_{m>0} \alpha_m^\mu \alpha_{-m}^\nu + 4\sum_{m>0} \tilde{\alpha}_m^\mu \tilde{\alpha}_{-m}^\nu \ . \tag{2.42}$$

2.2.5 Quantization

Canonical quantization

To carry on the procedure of canonical quantization, we need to compute some classical quantities. Starting from the expression of the Lagrangian:

$$\mathcal{L} = T\left(\dot{X}^2 - X'^2\right) \ , \tag{2.43}$$

we compute the momentum conjugate to X^μ, Π^μ:

$$\Pi^\mu = \frac{\delta\mathcal{L}}{\delta\dot{X}_\mu} \ . \tag{2.44}$$

So we have the classical Poisson brackets:

$$\left[\Pi^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\right]_{P.B.} = \left[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\right]_{P.B.} = 0 \ , \tag{2.45}$$

$$\left[\Pi^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\right]_{P.B.} = \eta^{\mu\nu}\delta(\sigma-\sigma') \ ; \tag{2.46}$$

where $\delta(\sigma-\sigma') = \sum_{m>0} e^{2im(\sigma-\sigma')}$.

If we substitute the mode expansions of the closed string into the classical Poisson brackets, we get the brackets for the modes:

$$[\alpha^\mu_m, \alpha^\nu_n]_{P.B.} = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n]_{P.B.} = 0 \ , \tag{2.47}$$

$$[\alpha^\mu_m, \tilde{\alpha}^\nu_n]_{P.B.} = im\eta^{\mu\nu}\delta_{m+n,0} \ . \tag{2.48}$$

And for the Virasoro operators we find:

$$[L_m, L_n]_{P.B.} = i(m-n)L_{m+n} \ . \tag{2.49}$$

We can now quantize the theory by replacing Poisson brackets with commutators

$$[\ldots,\ldots]_{P.B.} \to i[\ldots,\ldots] \ . \tag{2.50}$$

We rewrite the modes as:

$$\begin{cases} a^\mu_m = \frac{1}{\sqrt{m}}\alpha^\mu_m \\ \tilde{a}^\mu_m = \frac{1}{\sqrt{m}}\tilde{\alpha}^\mu_m \end{cases} , \tag{2.51}$$

having:

$$\left(a^\mu_m\right)^\dagger = a^\mu_{-m} \ . \tag{2.52}$$

The algebra of these new operators is:

$$[a^\mu_m, a^\nu_n] = [\tilde{a}^\mu_m, \tilde{a}^\nu_n] = 0 \ , \tag{2.53}$$

$$[a^\mu_m, \tilde{a}^\nu_n]_{P.B.} = \eta^{\mu\nu}\delta_{m+n,0} \ . \tag{2.54}$$

It is easy to see that the algebra satisfied by these new operators is essentially that of raising and lowering operators. So, we can interpret them as creation and annihilation operators.

However, a problem arises when considering states created by an odd number of zero mode operators $|\varphi\rangle = a_1^{0\dagger} a_2^{0\dagger} \dots a_{odd}^{0\dagger} |0\rangle$ because when we consider their norm $\langle \varphi \, |\varphi\rangle$ (using the commutators to evaluate the action of the operators acting on the vacuum) we get a negative result. An infinite tower of negative norm states arises in this way. We can tackle this problem by using the infinite number of constraints on L_n and $\tilde{L}_n$, which we write as:

$$L_n \, |\varphi\rangle = 0 \ ; \tag{2.55}$$

with $L_n^\dagger = L_{-n}$ we also have:

$$\langle\varphi| \, L_{-n} = 0 \ . \tag{2.56}$$

The same will be true for $\tilde{L}_n$.

Since in canonical quantization the quantum operators are derived from the normal ordered classical operators, we have another issue regarding the ordering of Virasoro operators. Indeed, for the zero mode L_0 and $\tilde{L}_0$ we have an order ambiguity:

$$L_0 = \frac{1}{2} \sum_{n=-\infty}^{+\infty} : \alpha^\mu_n \alpha^\nu_{-n} : \ . \tag{2.57}$$

To normal order the operator we will be forced to introduce a constant a so that:

$$(L_0 - a) \, |\varphi\rangle = 0 \ , \tag{2.58}$$

the same applies to $\tilde{L}_0$.

Furthermore, we have to impose the so called **level matching condition**:

$$\left(L_0 - \tilde{L}_0\right)|\varphi\rangle = 0\ . \tag{2.59}$$

Light-cone quantization

In light-cone coordinates we rearrange the components of the scalar fields:

$$X^\mu = \left(X^0, X^1, \dots X^{D-1}\right) \to \left(X^+, X^-, X^1, \dots X^{D-2}\right)\ ,$$

where $X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right)$.

The space-time metric becomes

$$\eta = \begin{pmatrix} 0 & -1 & \\ -1 & 0 & \\ & & \\ & & 1_{D-2\times D-2} \\ & & \end{pmatrix}\ .$$

So, for the most general expression of X^+ we can write:

$$X^+(\tau,\sigma) = x^+ + l_s^2 p^+\left(\tau + \frac{il_s}{p^+}\sum \frac{1}{m}\alpha_m^+ e^{-im\tau} cos\,(m\sigma)\right)\ . \tag{2.60}$$

We now use conformal invariance of the action to make the reparametrization

$$\begin{cases} \sigma_+ \to \tilde{\sigma}_+ = f_+(\sigma_+) = \tilde{\tau} + \tilde{\sigma}_+ \\ \sigma_- \to \tilde{\sigma}_- = f_-(\sigma_-) = \tilde{\tau} - \tilde{\sigma}_- \end{cases}\ , \tag{2.61}$$

where $\tilde{\tau} = \frac{1}{2}\left(f_+(\sigma_+) + f_-(\sigma_-)\right)$ and it must hold: $\partial_+\partial_-\tilde{\tau} = 0$.

Any transformation which gives a $\tilde{\tau}$ that satisfies the above equation is a symmetry of the theory.

The oscillator modes in (2.60) do satisfy that equation, so we set:

$$\tilde{\tau} = \tau + \frac{il_s}{p^+}\sum \frac{1}{m}\alpha_m^+ e^{-im\tau} cos\,(m\sigma)\ ; \tag{2.62}$$

which gives:

$$X^+ = x^+ + l_s^2 p^+ \tilde{\tau}\ , \tag{2.63}$$

$$\dot{X}^+ = \partial_{\tilde{\tau}} X^+ \ , \tag{2.64}$$

$$X^{+'} = \partial_\sigma X^+ = 0 \ . \tag{2.65}$$

We rewrite the first constraint in (2.26):

$$\dot{X} \cdot X' = 0 \to -\dot{X}^+ \cdot X^{-'} - \dot{X}^- \cdot X^+ + \dot{\vec{X}} \cdot \vec{X}' = 0 \ ,$$

$$-\left(l_s^2 p^+\right) X^{-'} + \dot{\vec{X}} \cdot \vec{X}' = 0 \ ,$$

$$X^{-'} = \frac{1}{(l_s^2 p^+)} \sum_{i=1}^{D-2} \dot{X}^i X^{i'} \ . \tag{2.66}$$

We also have that $X^{-'} = -l_s \sum_{m \neq 0} \alpha_m^- e^{-im\tau} sin\,(m\sigma)$. So we can find the expression on the modes α_m^- by setting $\tilde{\tau} = 0$ and integrating. We obtain:

$$\alpha_n^- = \frac{1}{l_s p^+} \left(\frac{1}{2} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \alpha_{n-m}^i \alpha_m^i \right) \ . \tag{2.67}$$

In this way we have expressed the $\pm$ quantities in terms of the $D-2$ transverse vector ones i.

All the states of the theory (the Hilbert space) can be constructed with the transverse modes α^i only: $|\varphi\rangle = a_{m_1}^{i_1\dagger} a_{m_2}^{i_2\dagger} \dots a_{m_r}^{i_r\dagger} |0\rangle$.

In this way negative-norm states do not arise! The downside of this approach is that Lorentz invariance is not manifest (as i goes from 1 to $D-2$) and we must check it explicitely.

Open string spectrum Let's take a look at the spectrum of the theory. In order to do that, we have to find the expression for the mass of the states.

First, we compute the following expression: $2p^+p^- = \sum_{i=1}^{D-2} p^i p^i + \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} : \alpha_{-m}^i \alpha_m^i : -a$, where a is a constant to be determined.

The mass is then given by:

$$\alpha' M^2 = -p^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} : \alpha_{-m}^i \alpha_m^i : -a \ . \tag{2.68}$$

We can define the number operator N :

$$N \equiv \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \alpha_{-m}^{i} \alpha_{m}^{i} = \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} m \left(a_{m}^{i\dagger} a_{m}^{i} \right) \ ; \tag{2.69}$$

and rewrite the mass squared operator as:

$$\alpha' M^2 = N - a \ . \tag{2.70}$$

The spectrum is then:

- $N = 0$: $\alpha' M^2 \left|0\right\rangle = -a \left|0\right\rangle$;
- $N = 1$: $\alpha' M^2 a_1^{i\dagger} \left|0\right\rangle = 1 - a \left|\varphi\right\rangle^i$
  It is a vector representation with $D-2$ degrees of freedom which means it is a massless vector. So: $1 - a = 0 \rightarrow a = 1$;
- $N = 2$: $\alpha' M^2 a_2^{i\dagger} \left|0\right\rangle = 2 - a \left|0\right\rangle$ and $\alpha' M^2 a_1^{i\dagger} a_1^{j\dagger} \left|0\right\rangle = 2 - a \left|0\right\rangle$.

After having determined the constant a, we realize that we have an issue: the spectrum contains a **tachyon**, a particle with negative mass!

Closed string spectrum For the closed string the treatment is similar, but here we have two separate copies of the oscillator modes that gives different operators.

We have two number operators N and $\tilde{N}$:

$$N = \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} m \left(a_{m}^{i\dagger} a_{m}^{i} \right) \ , \tag{2.71}$$

$$\tilde{N} = \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} m \left(\tilde{a}_{m}^{i\dagger} \tilde{a}_{m}^{i} \right) \ . \tag{2.72}$$

The mass operator is then given by

$$\alpha' M^2 = N + \tilde{N} - a \ . \tag{2.73}$$

We have the following spectrum (with $a = 1$) :

- $N + \tilde{N} = 0$: $\alpha' M^2 = -1$,
  tachyon;

- $N+\tilde{N}=1$: $\alpha' M^2 = 0$,
  graviton, 2-form, dilaton.

2.3 Superstring

So far we have only been talking about the bosonic string theory but this can't be a satisfactory model of nature as it lacks fermions. If we want to include fermions in string theory we also need supersymmetry. We will only consider the Ramond-Neveu-Schwarz (RNS) formalism which is based on world-sheet supersymmetry. In the RNS formalism the fermionic field $\psi^\mu(\sigma,\tau)$ is introduced, this is the superpartner of the $X^\mu(\sigma,\tau)$ field. The field ψ^μ is a two component spinor on the world-sheet and a vector under Lorentz transformations of the background space-time.

2.3.1 Superstring action

The complete action is given by the one for D massless bosons plus the standard Dirac action for D massless fermions:

$$S = -\frac{1}{2\pi\alpha'}\int d^2\sigma \left(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right) \qquad \alpha = 0,1 \,, \tag{2.74}$$

$\bar{\psi} = i\psi^\dagger \rho^0$ is the Dirac conjugate of a spinor and ρ^α represent the 2-dimensional Dirac matrices satisfying:

$$\left\{\rho^\alpha, \rho^\beta\right\} = 2\eta^{\alpha\beta} \,. \tag{2.75}$$

For example, one possible choice for these matrices is:

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,. \tag{2.76}$$

We can impose Majorana or Weyl conditions or both of them to the spinors.

- Majorana condition: $\psi^\dagger = \psi$,
  the spinor has real components;

- Weyl condition: $\rho\psi = \pm\psi$ where $\rho = \rho^0\rho^1$,
  which means we have two possible chiralities for the spinor.

Light-cone coordinates

Using light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$, the fermionic fields take the form:

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} , \tag{2.77}$$

where $\pm$ subscripts indicate the chirality of the spinor component. The fermionic part of the action in light-cone coordinates is:

$$S_\psi = \frac{1}{2\pi\alpha'} \int d\sigma_+ d\sigma_- \left[\psi_+ \cdot (\partial_- \psi_+) + \psi_- \cdot (\partial_+ \psi_-)\right] . \tag{2.78}$$

World-sheet supersymmetry

The complete action is invariant under the supersymmetry transformations, which mix the bosonic and fermionic fields:

$$\delta X^\mu = \bar{\epsilon}\psi^\mu , \tag{2.79}$$

$$\delta\psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon , \tag{2.80}$$

where ϵ is a constant (when we consider global supersymmetry transformaztions) infinitesimal Majorana spinor with two components:

$$\epsilon = \begin{pmatrix} \epsilon_- \\ \epsilon_+ \end{pmatrix} . \tag{2.81}$$

2.3.2 Equations of motion

In order to find the equations of motions for the fermioni fields we vary the action and set its variation to zero:

$$\delta S_\psi = 0 \ \rightarrow \delta \mathcal{L}_\psi = 0 .$$

The equations of motions for the fermionic fields, together with the ones of the bosonic fields X^μ, are the equations of motion of superstring theory. In terms of the two spinor components, using light-cone coordinates, the equations of motion for the fermionic fields are:

$$\delta S_\psi = \int d^2\sigma \left[\delta\psi_+ \cdot (\partial_-\psi_+) + \psi_+ \cdot \delta\left(\partial_-\psi_+\right) + \psi_- \cdot \delta\left(\partial_+\psi_-\right) + \delta\psi_- \cdot (\partial_+\psi_-) +\right] =$$

$$= \int d^2\sigma \left[2\delta\psi_+ \cdot (\partial_-\psi_+) + 2\delta\psi_- \cdot (\partial_+\psi_-) + \partial_-\left(\psi_+ \cdot \delta\psi_+\right) + \partial_+\left(\psi_- \cdot \delta\psi_-\right)\right] \ , \qquad (2.82)$$

where we have integrated by parts: $\psi_+ \cdot \partial_-\delta\psi_+ = -\partial_-\psi_+ \cdot \delta\psi_+ + \partial_-\left(\psi_+ \cdot \delta\psi_+\right)$.

If we take the total derivative terms to vanish (we will look at suitable boundary conditions later), we obtain the following Dirac equations as equations of motion:

$$\partial_+\psi_- = 0 \ , \qquad \partial_-\psi_+ = 0 \ , \qquad (2.83)$$

which describe right-moving and left-moving waves.

2.3.3 Boundary conditions

The boundary terms are:

$$\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\sigma - \partial_\sigma\left(\psi_+ \cdot \delta\psi_+\right) + \partial_\sigma\left(\psi_- \cdot \delta\psi_-\right) =$$

$$= \int_{-\infty}^{\infty} d\tau - \left(\psi_+ \cdot \delta\psi_+|_{\sigma=\pi} - \psi_- \cdot \delta\psi_-|_{\sigma=\pi}\right) + \left(\psi_+ \cdot \delta\psi_+|_{\sigma=0} - \psi_- \cdot \delta\psi_-|_{\sigma=0}\right) \ , \qquad (2.84)$$

where the derivatives with respect to τ vanish because the fields vanish at $\pm\infty$.

The two terms must vanish independently and we have two possible boundary conditions.

For **open strings** the two terms must vanish independetly for each endpoint of the string. This happens when $\psi_+ = \pm\psi_-$. Since the relative sign is a matter of convention, at one end we are free to impose:

$$\psi_+^\mu|_{\sigma=0} = \psi_-^\mu|_{\sigma=0} \ . \qquad (2.85)$$

At the other end we now have two possibilities:

- Ramond boundary condition

$$\psi_+^\mu|_{\sigma=\pi} = \psi_-^\mu|_{\sigma=\pi} \ ; \qquad (2.86)$$

which gives the mode expansions for left and right movers:

$$\psi_{-}^{\mu}(\sigma.\tau) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^{\mu} e^{-ir(\tau-\sigma)} \ , \tag{2.87}$$

$$\psi_{+}^{\mu}(\sigma.\tau) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^{\mu} e^{-ir(\tau+\sigma)} \ . \tag{2.88}$$

- Neveu-Schwarz boundary condition

$$\psi_{+}^{\mu}|_{\sigma=\pi} = -\psi_{-}^{\mu}|_{\sigma=\pi} \ . \tag{2.89}$$

With mode expansions:

$$\psi_{-}^{\mu}(\sigma.\tau) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^{\mu} e^{-in\sigma_{-}} \ , \tag{2.90}$$

$$\psi_{+}^{\mu}(\sigma.\tau) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^{\mu} e^{-in\sigma_{+}} \ . \tag{2.91}$$

The Majorana condition also requires:

$$d_{-n}^{\mu} = d_n^{\mu\dagger} \ , \qquad b_{-r}^{\mu} = b_r^{\mu\dagger} \ . \tag{2.92}$$

For **closed strings** two possible periodicity conditions make the boundary term vanish:

$$\psi_{\pm}(\sigma,\tau) = \pm\psi_{\pm}(\sigma+\pi,\tau) \ ; \tag{2.93}$$

where the upper sign stand for periodic boundary condition and the lower sign for antiperiodic boundary conditions.

Which give the mode expansions

$$\psi_{-}^{\mu}(\sigma.\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^{\mu} e^{-2ir(\tau-\sigma)} \ , \qquad \psi_{+}^{\mu}(\sigma.\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^{\mu} e^{-2ir(\tau+\sigma)} \ , \tag{2.94}$$

$$\psi_{-}^{\mu}(\sigma.\tau) = \sum_{n\in\mathbb{Z}} d_n^{\mu} e^{-2in(\tau-\sigma)} \ , \qquad \psi_{+}^{\mu}(\sigma.\tau) = \sum_{n\in\mathbb{Z}} d_n^{\mu} e^{-2in(\tau+\sigma)} \ . \tag{2.95}$$

We can impose periodicity (Ramond) or antiperiodicity (Neveu-Schwarz) boundary conditions for right movers and left movers independently. Therefore, we have four possible choices on how to combine boundary conditions: R-R, R-NS, NS-R, NS-NS.

Actually, there are only two possible inequivalent theories combining the different sectors and they are classified by the choice of the R vacuum state. In each of the four secors we have 64 states.

IIB

R-R: $|+\rangle_R \otimes |+\rangle_R$,
gives a scalar λ, a two-form gauge field $A_{\mu\nu}$ and a four-form gauge field $D_{\mu\nu\rho\sigma}$ whose field strength is self dual;

NS-R: $\tilde{b}^i_{-\frac{1}{2}} |0\rangle_{NS} \otimes |+\rangle_R$;

R-NS: $|+\rangle_R \otimes b^i_{-\frac{1}{2}} |0\rangle_{NS}$,
the two mixed sectors together give two spin $\frac{3}{2}$ fermions ${\psi_+}^i{}_\mu$ and two spin $\frac{1}{2}$ fermions χ^i_+ ;

NS-NS: $\tilde{b}^i_{-\frac{1}{2}} |0\rangle_{NS} \otimes b^i_{-\frac{1}{2}} |0\rangle_{NS}$,
gives a scalar ϕ, an antisymmetric two-form gauge field $B_{\mu\nu}$ and a symmetric traceless rank-two tensor $G_{\mu\nu}$;

IIA

R-R: $|-\rangle_R \otimes |+\rangle_R$,
gives a one-form gauge field A_μ and a three-form gauge field $A_{\mu\nu\rho}$;

NS-R: $\tilde{b}^i_{-\frac{1}{2}} |0\rangle_{NS} \otimes |+\rangle_R$;

R-NS: $|-\rangle_R \otimes b^i_{-\frac{1}{2}} |0\rangle_{NS}$,
the two mixed sectors together give two spin $\frac{3}{2}$ fermions ${\psi_+}^i{}_\mu$, ${\psi_-}^i{}_\mu$ and two spin $\frac{1}{2}$ fermions χ^i_+ χ^i_+, χ^i_-;

NS-NS: $\tilde{b}^i_{-\frac{1}{2}} |0\rangle_{NS} \otimes b^i_{-\frac{1}{2}} |0\rangle_{NS}$,
gives a scalar ϕ, an antisymmetric two-form gauge field $B_{\mu\nu}$ and a symmetric traceless rank-two tensor $G_{\mu\nu}$;

2.3.4 Canonical quantization

To quantize the theory we need to know the canonical anticommutation relations for the fermionic world-sheet fields ψ^μ in addition to the commutation relations for the bosonic fields X^μ. They are:

$$\{\psi_A^\mu(\sigma,\tau), \psi_B^\nu(\sigma',\tau)\} = \eta^{\mu\nu}\delta_{AB}\delta(\sigma-\sigma') \ . \tag{2.96}$$

Now, substituting the mode expansions for the fermionic fields we find:

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \ , \tag{2.97}$$

for the NS sector;

$$\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0} \ , \tag{2.98}$$

for the R sector.

If we consider the zero-components of the modes ($\mu,\nu=0$) we still find negative norm states appearing from the fermionic fields.

Another intersting fact is that if we look at the zero modes ($m,n=0$) of the R sector we see that could rescale the modes d_m^μ so that they give a Clifford algebra $\{\Gamma^\mu,\Gamma^\nu\}=2\eta^{\mu\nu}$. This is not possible in the NS sector because $r,s \geq \frac{1}{2}$. The states created by those zero modes $d_0^\mu|0\rangle$ will then be space-time fermions and we will have $2^{\frac{D}{2}}$ of them.

Mass spectrum

Let's consider the possibilities in the case of open strings, since we have two possible boundary conditions, we will have two different sectors:

- NS sector

The mass squared operator in light-cone coordinates, before normal ordering, is:

$$M^2 = \frac{1}{\alpha'}\sum_{i=1}^{D-2}\left(\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^i b_r^i - a_{NS}\right) \ , \tag{2.99}$$

where we can define $N_{NS} = \sum_{i=1}^{D-2}\left(\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^i b_r^i\right)$.

The spectrum is then:

$N_{NS} = 0$**:** $\alpha' M^2 |0\rangle = -a_{NS} |0\rangle$;

$N_{NS} = \frac{1}{2}$**:** $\alpha' M^2 b^{i\dagger}_{\frac{1}{2}} |0\rangle = \frac{1}{2} - a_{NS} |\varphi\rangle^i$,

It is a vector representation with $D-2$ degrees of freedom which means it is a massless vector. So: $\frac{1}{2} - a_{NS} = 0 \rightarrow a_{NS} = \frac{1}{2}$.

- R sector

The mass squared operator is:

$$M^2 = \frac{1}{\alpha'} \sum_{i=1}^{D-2} \left(\sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n + \sum_{n=1}^{\infty} n d^i_{-n} d^i_n - a_R \right) \ , \tag{2.100}$$

where we can define: $N_R = \sum_{i=1}^{D-2} \left(\sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n + \sum_{n=1}^{\infty} n d^i_{-n} d^i_n \right)$.

$N_R = 0$**:** $\alpha' M^2 |0\rangle = -a_R |0\rangle$,

we found the vacuum to be a massless spinor, so $a_R = 0$.

For closed strings we have to consider the interplay between left-movers $(-)$ and right-movers $(+)$. There are four possibilities: R-R, R-NS, NS-R, NS-NS .

The mass-square operator is in any case:

$$M^2 = \frac{1}{\alpha'} \left(M_L^2 + M_R^2 \right) \ , \tag{2.101}$$

where M_L^2 and M_R^2 are the mass-squared operators of open strings used for the left and right sector rispectively.

Giolizzi, Scherk, Olive (GSO) projection

We want the degrees of freedom of the R sector to match the ones in the NS sector, so we introduce the following operators:

$$G_{NS} = -\left(-1\right)^{F_{NS}} \ , \tag{2.102}$$

where $F_{NS} = \sum_{i=1}^{D-2} \sum_{r=\frac{1}{2}}^{\infty} b^i_{-r} b^i_r$ counts the number of b in a state.

$$G_R = \Gamma_{11} \left(-1\right)^{F_R} \ , \tag{2.103}$$

where $F_r = \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} d^i_{-n} d^i_n$ counts the number of d in a state.

The GSO projection consists in keeping only that states, regarding them as physical states, which have:

$$G_{NS}\,|\varphi\rangle = +\,|\varphi\rangle\ \ , \tag{2.104}$$

$$G_{R}\,|\varphi\rangle = +\,|\varphi\rangle\ \ . \tag{2.105}$$

Through the GSO projection we can eliminate some of the states of the spectrum. The operator used for the GSO projection is called the G-parity operator. This operator has two possibile eigenvalues: ± 1. States which have eigenvalue $+1$ are said to have positive G-parity and states with eigenvalue -1 negative G-parity. We can then truncate the spectrum in a way that is consistent with supersymmetry by keeping only states which have positive or negative G-parity in a certain sector (R or NS).

For example, we have type IIA and type IIB superstring theories which differ only by the G-parity of the R-sectors. In type IIA the two R-sectors (for left and right movers) are taken with opposity G-parity while in type IIB they have the same G-parity. This gives rise to different spectra even if the two theories are both born as superstring theories.

2.3.5 D-branes

D-branes (or more generally Dp-branes, where p denotes the number of spatial dimensions of the brane) are p dimensional extended objects where the endpoints of open strings can end. A D0-brane is a point, a D1-brane is a string, a D2-brane is a plane and so on.

The electromagnetic example

We look at the case of electromagnetism in four dimensional space to get interesting hints that can be generalized to our string theory framework.

The Maxwell equations in vacuum are:

$$d^{*}F = 0 \quad , \quad dF = 0\ . \tag{2.106}$$

In the presence of an electric charge we have to introduce a current j_e such that:

$$d^{*}F = j_e \quad , \quad dF = 0\ , \tag{2.107}$$

and we can write $F = dA$.

Equivalently, if there is a magnetic charge (magnetic monopole) we introduce a current j_m and the Maxwell equations become:

$$d^*F = 0 \quad , \quad dF = j_m \ , \tag{2.108}$$

defining $\tilde{F} =^* F$, we can write $\tilde{F} = d\tilde{A}$,

and we can see that this magnetic case is dual to the previous electric case.

If we consider electromagnetism in D dimensions, things change slightly.

For an electric charge (point-like) we still have a gauge potential A which is a one-form and a two-form field strenght $F = dA$, while the dual field strength $\tilde{F}$ is now a $(D-2)$-form. This gives a $(D-3)$-form gauge potential $\tilde{A}$, which means the magnetic charge is not a point-like.

In general, when we have a $(1+p)$-form gauge potential, it will couple electrically to a p-dimensional object and magnetically to a $D-(p+4)$ dimensional object. One can see this from this chain of relations:

$$p - brane \to A_{1+p} \to F_{2+p} \ ,$$

$$^*F_{2+p} \to \tilde{F}_{D-(2+p)} \to \tilde{A}_{D-(p+3)} \to (D-(p+4)) - brane \ .$$

So, in the case of the two type II superstring theories we can find the branes that couple to the forms we have in the theories.

IIB

R-R:

- A_0: has a (-1)-brane (instanton) electric charge and a 7-brane magnetic charge;
- A_2: has a 1-brane electric charge and a 5-brane magnetic charge;
- A_4: has a 3-brane electric charge and a 3-brane magnetic charge;

NS-NS:

- B_2: has a 1-brane electric charge (the fundamental string) and a 5-brane magnetic charge (NS 5-brane).

IIA

R-R:

- A_1: has a 0-brane electric charge and a 6-brane magnetic charge;
- A_3: has a 2-brane electric charge and a 4-brane magnetic charge;

NS-NS:

- B_2: has a 1-brane electric charge (the fundamental string) and a 5-brane magnetic charge (NS 5-brane).

3 Supersymmetry and supergravity

3.1 Supersymmetry

In supersymmetric theories we postulate the existence of a symmetry between fermions and bosons, called supersymmetry, and we build Lagrangians which are invariant under supersymmetry transformations of the fields. Those transformations are generated by a fermionic operator Q (we know it is fermionic because it changes sign under 360 degree rotations) which turns fermionic states into bosonic ones and vice versa.

This operator is related to space-time symmetries (since the spin is connected with spatial rotations) and to internal symmetries. The interesting thing is that by performing two successive supersymmetry transformations on a state we get back the same state but evaluated at a different space-time position. This is how supersymmetry is linked to space-time symmetries. Also, when making supersymmetry a local symmetry of the theory, we get fields that reproduce General Relativity and the theory we obtain is called Supergravity.

Let's see how supersymmetry is actually implemented in a theory. Bosonic generators of symmetry transformations form a Lie algebra, so we have to extend this concept in order to accomodate fermionic generators too.

3.1.1 Lie superalgebra

The definition for an ordinary Lie algebra g is:

1. g is a vector space on $\mathbb{R}$ (or $\mathbb{C}$);

2. there is an internal composition law $[\cdot,\cdot]$ which is bilinear and antisymmetric;

3. the Jacobi identity holds:

$$[[A,B],C]+[[B,C],A]+[[C,A],B]=0\ . \tag{3.1}$$

In a Lie superalgebra we have both bosons and fermions, we assign a certain grade to them to distinguish the different kind of operators. Bosons have grade 0 (even) while fermions have grade 1 (odd). So, the definition of a superalgebra s is:

1. s is a graded vector space on $\mathbb{C}$: $s =^0 s\ \cup\ ^1 s$;

2. there is an internal composition law $[\cdot,\cdot]$ which is bilinear and superantisymmetric (symmetric for fermions and antisymmetric for bosons):

$$[A,B]=(-1)^{1+grad(A)grad(B)}\,[B,A]\ ; \tag{3.2}$$

   and it is additive with respect to the grade:

$$C=[A,B]\ \rightarrow\ grad\,(C)=grad\,(A)+grad\,(B)\ ; \tag{3.3}$$

3. the superJacobi identity holds

$$(-1)^{grad(A)grad(C)}\,[[A,B],C]+(-1)^{grad(B)grad(A)}\,[[B,C],A]+(-1)^{grad(C)grad(B)}\,[[C,A],B]=0\ . \tag{3.4}$$

We have introduced the concept of Lie superalgebra because it is what is mathematically needed to be able to construct an algebra which mixes space-time symmetries (P) with internal symmetries (g) non-trivially (which means differently from $P\oplus g$).

3.1.2 Supersymmetry algebra

The bosonic part of the algebra, which consists of the Poincaré algebra and the internal symmetry algebra, has generators $P_\mu,\ J_{\mu\nu}\ \in P$ and $T_r\in g$, with commutation rules:

$$[P_\mu,P_\nu]=0\ , \tag{3.5}$$

$$[P_\mu,J_{\nu\rho}]=\eta_{\mu\nu}P_\rho-\eta_{\mu\rho}P_\nu\ , \tag{3.6}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\sigma} J_{\nu\rho} + \eta_{\nu\rho} J_{\mu\sigma} - (\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho}) \ , \tag{3.7}$$

$$[P_\mu, T_r] = 0 \ , \tag{3.8}$$

$$[J_{\mu\nu}, T_r] = 0 \ , \tag{3.9}$$

$$[T_r, T_s] = f_{rst} T_t \ . \tag{3.10}$$

We call the fermionic generator Q^i_α, where $i = 1, \ldots, N$ is the index of a representation of g (we will have N such generators called supercharges) and α is a spinorial index (spin $\frac{1}{2}$). It has the following commutation and anticommutation rules:

$$\left[Q^i_\alpha, P_\mu\right] = 0 \ , \tag{3.11}$$

$$\left[Q^i_\alpha, J_{\mu\nu}\right] = \frac{1}{2} \left(\gamma_{\mu\nu}\right)^\beta_\alpha Q^i_\beta \ ; \tag{3.12}$$

where $\gamma_{\mu\nu} = \frac{1}{2}\left(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu\right)$ and γ_μ are Dirac matrices which satisfy: $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$.

If we consider a theory where $N = 1$, we find:

$$\{Q_\alpha, Q_\beta\} = 2 \left(\gamma_\mu C\right)_{\alpha\beta} P_\mu \ ; \tag{3.13}$$

where $C_{\alpha\beta}$ is the charge conjugation matrix, defined through the relation $C\gamma^\mu C^{-1} = -\left(\gamma^\mu\right)^T$.

and finally:

$$[Q_\alpha, T_r] = i \left(\gamma_5\right)^\beta_\alpha Q_\beta t_r \ ; \tag{3.14}$$

where t_r is an arbitrary complex number which can be determined using the Jacobi identity for Q_α, T_r and T_s. We then find that we can set it to one.

3.1.3 Representation of N=1 SUSY algebra

To find the representations of the SUSY algebra, we can use Wigner's method of reduced representations. We set the reference momentum to q^μ and we find the unitary representations of the subgroup which leaves q^μ invariant (little group of q^μ).

- Massless case: $q^\mu q_\mu = 0$.

Using the SUSY algebra we find that the Q_α form a Clifford algebra of raising/lowering operators for the helicity of a state. Q raises the helicity by $\frac{1}{2}$ while $Q^\dagger$ lowers it by the same amount.

So, if we start from a state with maximum helicity λ, we can find the other states of the same multiplet by acting with the lowering operator. In the case of $N = 1$ we have the multiplet:

$$|\lambda\rangle \ , \tag{3.15}$$

$$\left|\lambda - \frac{1}{2}\right\rangle = Q^\dagger \, |\lambda\rangle \ . \tag{3.16}$$

We also have to couple the state $|\lambda\rangle$ with the state of opposite helicity $|-\lambda\rangle$ because of CPT invariance. So we add the CPT conjugate to the multiplet we have found:

$$|-\lambda\rangle \ , \tag{3.17}$$

$$\left|-\lambda + \frac{1}{2}\right\rangle = Q \, |-\lambda\rangle \ . \tag{3.18}$$

At the end, we have different multiplets for the various possible values of the helicity of the starting state (λ):

$$\lambda = \frac{1}{2} : \qquad \left|\frac{1}{2}\right\rangle , |0\rangle , |0\rangle , \left|-\frac{1}{2}\right\rangle \ ;$$

the $\lambda = \frac{1}{2}$ multiplet contains two scalars $|0\rangle$ and two states of a spin $\frac{1}{2}$ fermions $\left|\frac{1}{2}\right\rangle$.

$$\lambda = 1 : \qquad |1\rangle , \left|\frac{1}{2}\right\rangle , \left|-\frac{1}{2}\right\rangle , |-1\rangle \ ;$$

the $\lambda = 1$ multiplet contains to two states of a vector $|1\rangle$ and two states of a spin $\frac{1}{2}$ fermions $\left|\frac{1}{2}\right\rangle$.

$$\lambda = \frac{3}{2} : \qquad \left|\frac{3}{2}\right\rangle , |1\rangle , |-1\rangle , \left|-\frac{3}{2}\right\rangle \ ;$$

the $\lambda = \frac{3}{2}$ multiplet contains two states of a vector $|1\rangle$ and two states of a spin $\frac{3}{2}$ fermions $\left|\frac{3}{2}\right\rangle$, the gravitino.

$$\lambda = 2 : \qquad |2\rangle , \left|\frac{3}{2}\right\rangle , \left|-\frac{3}{2}\right\rangle , |-2\rangle \ ;$$

the $\lambda = 2$ multiplet contains the two polarizations of the graviton $|2\rangle$ and two states of a spin $\frac{3}{2}$ fermions $\left|\frac{3}{2}\right\rangle$, the gravitino.

- Massive case: $q^\mu q_\mu = -m^2$, $\qquad q^\mu = (m, 0, 0, 0)$.

Since rotations leave q^μ invariant we have that the little group is $SO(3)$. The Q algebra is:

$$\left\{Q^i_A, \left(Q^B_j\right)^*\right\} = 2m\delta^B_A\delta^i_j \ , \tag{3.19}$$

$$\left\{Q^i_A, Q^B_j\right\} = \left\{\left(Q^i_A\right)^*, \left(Q^B_j\right)^*\right\} = 0 \ . \tag{3.20}$$

All the Q contribute to enlarge the multiplet as they form a algebra of raising/lowering operators. As before, the total number of states in a multiplet is 2^{2N}.

Since every state has fixed spin, when we lower the spin we mix states with different spin (we get all possible states from $|l - s|$ to $|l + s|$). Although, in this way it is not clear from the structure of the multiplet what is its particle content.

3.1.4 Supersymmetry transformations for N=1 SUSY

Let's look at the SUSY transformations for the symplest supersymmetric model with $N = 1$ which is composed of two free (non-interacting) massless fields. One is a complex scalar field ϕ and the other a Weyl spinor of spin $\frac{1}{2}$ χ_α (which we take to be left-chiral).

We consider a global transformation proportional to an infinitesimal fermionic parameter ϵ (which is space-time independent) which is a Weyl spinor of dimension $-\frac{1}{2}$. The postulate of supersymmetry is that supersymmetric transformations turn bosons into fermions and vice versa. So, by dimensional analysis we guess:

$$\delta\phi \simeq \epsilon \cdot \chi \ , \tag{3.21}$$

$$\delta\chi \simeq C\epsilon\partial_\mu\phi^\dagger \ . \tag{3.22}$$

To determine the actual transformations we must also check the behaviour of the guessed terms under Lorentz transformations. Eventually, we get:

$$\delta\phi = \epsilon \cdot \chi \ , \tag{3.23}$$

$$\delta\chi = -C^* \left(\partial_\mu \phi\right) \sigma^\mu i \sigma^2 \epsilon^* \ ; \tag{3.24}$$

where $\sigma^\mu = (1, \vec{\sigma})$, $\vec{\sigma}$ are the Pauli matrices and $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

3.2 Supergravity

When we consider local SUSY transformations for the fields, which means we are taking the parameter ϵ to be space-time dependent, we are forced to introduce a gauge field that has the properties of the graviton. For this reason, theories that exhibit local supersymmetry invariance are called supergravity theories.

The gauge multiplet consists of the frame field $e_\mu^a(x)$ which describes the graviton and N vector-spinor fields $\Psi_\mu(x)$ whose quanta are the gravitinos. In the case of $N = 1$ we only have the graviton and a Majorana spinor gravitino in the multiplet.

3.2.1 Supergravity as the low-energy limit of superstring theory

In the spectrum of a string theory we always have a finite number of massless states and an infinite tower of massive states at a mass scale charachterized by the string tension. If we are only interested in studying string theory in the low-energy limit ($\alpha' \to 0$ and $T \to \infty$) we can forget about the massive states and neglect their contribution to the action. In this way, we write down an effective action which only includes the fields corresponding to the massless states. In principle, a low-energy effective action S_{eff} for the massless fields can be found by integrating out the massive fields from the classical exact action S. By doing so, no approximation would be introduced but then the effective action would be non-local and very complicated. Since not even the exact action to start with is known, the procedure of integrating the heavy fields is out of reach. So, we take a different approach: we study the string S matrix elements and construct a classical action for the massless fields that reproduces them. Furthermore, the leading terms of the effective action constructed in this way can be found by symmetry principles: gauge invariance and local supersymmetry. What we find is that a good approximation to string theory is given by a supergravity theory describing the interactions of the massless modes only.

It is not obvious from the start that this approach can be useful for analyzing non-perturbative features of string theory, as extrapolations from weak to strong coupling are usually beyond control. Although, if one considers only quantities that are "protected" by supersymmetry many properties of the theory can be discovered.

3.2.2 M-Theory and $D = 11$ supergravity

After the discovery of the five different ten-dimensional superstring theories, it was found that they were all related to one another through different dualities. Moreover, two of those superstring theories exhibit an eleventh dimension at strong coupling. This 11-dimensional limit, called M-theory, does not contain strings but other extended objects called membranes.

We can see 11-dimensional supergravity as the low-energy effective action of M-theory, and start its analysis by studying the properties of the corresponding supergravity theory. Note that $D = 11$ is the maximum dimension for a supergravity theory, we know this from the following argument: in eleven dimnension the Lorentz group is $SO(1, 10)$, so spinors have $2^{\left\lfloor \frac{d}{2} \right\rfloor} = 32$ components. In four dimensions each eleven dimensional spinor would correspond to eigth four dimensional ones. So, if we consider $N = 8$ supersymmetry, we can have helicities from -2 to $+2$ in the same supermultiplet. Interestingly, only for spin $\leq \frac{5}{2}$ consistent interactions terms can be written down.

Field content

In eleven dimensional supergravity we have only three different fields: the graviton, represented by the vielbein $e_M^A(x)$ where $A, B, ...$ are tangent (flat) space indices and $M, N, ...$ are base (curved) space indices; a Majorana gravitino ψ_M which is the gauge field for local supersymmetry; finally, by counting fermionic and bosonic degrees of freedom (which must be equal because of supersymmetry) we discover that we also have a 3-form potential A_{MNP} which represents a rank 3 antisymmetric tensor.

Action

We take as a postulate the presence in the action of the graviton and gravitino terms together with the covariant kinetic term for the 3-form potential. We will also have additional terms but we are going to neglect them. Anyway, guided by the requirement of invariance under A_3 gauge transformations, together with general coordinate invariance, local supersymmetry and dimensional analysis, we write the action as

$$S = \frac{1}{2k^2} \int d^{11}x \, e \left[e^{AM} e^{BN} R_{MNAB} - \bar{\psi}_M \gamma^{MNP} D_N \psi_P - \frac{1}{24} F^{MNPQ} F_{MNPQ} + ... \right] , \tag{3.25}$$

where R_{MNAB} is the Riemann tensor, F_{MNPQ} is the field strength of A_{MNP} ($F_{MNPQ} = 3\partial_{[M} A_{NPQ]}$), k is the 11-dimensional Newton constant and γ^{MNP} is the antisymmetrized product of gamma matrices $\gamma^{MNP} = \frac{1}{3!} \left(\gamma^{[M} \gamma^{N} \gamma^{P]} \right)$.

Supersymmetry transformations

The action is invariant under local supersymmetry transformations, which depend on an infinitesimal space-time dependent Grassman parameter $\epsilon\,(x)$ that transforms as a Majorana spinor. The SUSY transformations are:

$$\delta e_M^A = \frac{1}{2} \bar{\epsilon} \gamma^A \psi_M \ , \tag{3.26}$$

$$\delta A_{MNP} = -3 \bar{\epsilon} \gamma_{[MN} \psi_{P]} \ , \tag{3.27}$$

$$\delta \psi_M = D_M \epsilon + \frac{1}{12} \frac{1}{4!} \left(\gamma_M^{NPQR} F_{NPQR} - \frac{1}{2} \gamma^{PQR} F_{MPQR} \right) \ ; \tag{3.28}$$

where the Dirac matrices in curved space are $\gamma_M = e_M^A \gamma_A$ and the covariant derivative of the spinor parameter is:

$$D_M \epsilon = \partial_M \epsilon + \frac{1}{4} \omega_{MAB} \gamma^{AB} \epsilon \ ; \tag{3.29}$$

ω_{MAB} is the spin connection and it can be expressed in terms of the elfbein:

$$\omega_{MAB} = \frac{1}{2} \left(-\Omega_{MAB} + \Omega_{ABM} + \Omega_{BMA} \right) \ ; \tag{3.30}$$

where $\Omega_{MN}^A = 2\partial_{[N} e_{M]}^A$.

3.2.3 Type IIA supergravity

Type IIA supergravity is the supergravity theory one gets as the low-energy limit of type IIA superstring theory. This superstring theory arises from a specific combination of boundary conditions for the closed string. It is also possible to obtain it from M-theory through dimensional reduction, which means we take one of the spatial directions to be circular and keep only the zero modes of the Fourier expansions of the fields on this direction. By identifying the bosonic and fermionic degrees of freedom in the correct way (fields in eleven dimensions can be decompose in a specific way to give rise to the right fields in ten dimensions) we find the same field content as in superstring theory. As previously explained, the supergravity theory is then found by keeping only the massless modes of the corresponding superstring theory.

Field content

The bosonic fields of type IIA supergravity come from the massless modes in the R-R and the NS-NS sectors. They are the graviton (g_{MN}), the Kalb-Ramond field (B_{MN}), the dilaton (ϕ), a rank-1 antisymmetric tensor (A_M) and a rank-3 antisymmetrci tensor (A_{MNP}).

The fermionic fields arise from the NS-R and R-NS sector. They are: two Majorana-Weyl gravitino (ψ^a_M , a is a spinor index) with opposite chirality and two Majorana-Weyl dilatino (λ^a) also with opposite chirality.

Action

The bosonic part of the action is composed of the graviton term together with the ones of the dilaton and the gauge fields. We can write it as:

$$S_b = \frac{1}{2k^2}\int d^{10}x\sqrt{-g}\left[e^{-2\phi}\left(R + 4\partial^M\phi\partial_M\phi - \frac{1}{2}H^{MNP}H_{MNP}\right) - \frac{1}{4}F^{MN}F_{MN} - \frac{1}{4}F^{MNPQ}F_{MNPQ}\right] , \tag{3.31}$$

plus a Chern-Simons term which we do not quote here.

The fermionic part will then include the kinetic terms for the gravitinos and the dilatinos.

Equations of motion

We list here the equations of motion for the bosonic fields of the theory as written in [14]; they are:

$$R + 4\nabla^2\Phi - 4\left(\nabla\Phi\right)^2 - \frac{1}{2}|H|^2 = 0\ , \qquad (dilaton) \tag{3.32}$$

$$e^{-2\phi}\left(R_{MN} + 2\nabla_M\nabla_N\phi - \frac{1}{2}H_M^{\ PQ}H_{NPQ}\right) - \frac{1}{4}\sum_{p\geq 2}|F_p|^2_{MN} = 0\ , \qquad (Einstein) \tag{3.33}$$

$$d\left(e^{-2\phi} * H\right) + \frac{1}{2}\sum_{p\geq 2} F_{p-2}\wedge *F_p = 0\ , \qquad (B-field) \tag{3.34}$$

$$dF_p + H\wedge F_{p-2} = 0\ , \qquad (Bianchi) \tag{3.35}$$

$$* F_p + (-1)^{p(p+1)/2}\, F_{10-p} = 0\ ; \qquad (duality) \tag{3.36}$$

where $|F_p|^2_{MN} = \frac{1}{(p-1)!}F_M^{\ Q_1\ldots Q_{p-1}}F_{NQ_1\ldots Q_{p-1}}$.

These equations will be of paramount importance to us, since they are the starting point of our work.

4 Kaluza-Klein supergravity

Kaluza-Klein supergravity is the result of the mixing of the Kaluza-Klein idea for compactifications and supergravity theories. By using those ideas together, we could regard supergravity theories as fundamental and take them as the actual theories of our physical world. Then our most succesful description of reality, the Standard model, could just be seen as the result of a specific compactification scheme of a more fundamental supergravity theory.

4.1 Compactification

Since we will be dealing with compactifications, let us take a brief overlook at what this procedure is and how it works. In short, compactificatifying a dimension means we identify the boundaries of a non-compact dimension to obtain a compact one. The most simple example of this procedure is the circle, which can be seen as the compactification of an infinite line where we have identified points at a distance of $2\pi Rk$ where k is an integer number.

If we are dealing with several spatial dimensions we can compactify some of those dimension while keeping the rest of them untouched. In this way, we can study theories in many spatial dimensions and then compactify all but three of them to see if the theory studied resembles the one we are used to ordinarily.

Compactifying a spatial dimension means we take the fields to be periodic in that direction. Since they are periodic we can expand them in Fourier modes, called Kaluza-Klein modes, along that direction. As a result, we get an infinite number of modes in the other directions which are referred to as KK tower.

4.2 The Kaluza-Klein idea

Kaluza and Klein first had the idea to introduce another space dimension in order to unify the known forces of nature (gravity and electromagnetism at their time) of the observable four dimensional space-time into a gravity theory living in a higher dimensional one.

Their suggestion is that it is possible to obtain a theory of electromagnetism and gravity in four dimensional space-time from a theory of gravity in a five dimensional space-time by compactifying the fifth dimension to a circle. If the size of the compactified dimension is taken to be very small, then we could imagine it to be impossible for us to detect it with our experiments. In this way, we would effectively experience a four dimensional reality even if it actually is a five dimensional one. Let's see in more details how this works.

The starting point is hypothesizing a five dimensional metric g_{MN} :

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + e^{2\sigma} A_\mu A_\nu & e^{2\sigma} A_\mu \\ e^{2\sigma} A_\nu & e^{2\sigma} \end{pmatrix} \; ; \tag{4.1}$$

where $g_{\mu\nu}$ is the standard four dimensional Minkowski metric, A_μ is a four vector and σ is a scalar field. We can now use this metric to write down Einstein's equations in this case. Then, if one implements the hypothesys that none of the terms appearing in the definition of the metric g_{MN} depends on the fifth coordinates y (called "cylinder condition") $\frac{\partial g_{MN}}{\partial y} = 0$, we can get the field equations of four dimensional general relativity and electromagnetism from Einstein's equations in five dimensions and the equations for the scalar σ.

By using the cylinder condition one avoids the "issue" of having an infinite tower of modes coming from the compactified dimension, but the theory obtained in this way still has many problems to be solved before one can consider it a serious candidate for the description of our reality. Nevetheless, the Kaluza-Klein idea seems very promising as it can be applied in different context where it might give rise to a more realistic theory. For example, it provides a mechanism through which one can tackle the problem of surplus spatial dimensions in string and supergravity theories.

4.3 Kaluza-Klein supergravity theories

One of the most recent development of the Kaluza-Klein idea is Kaluza-Klein supergravity. With the discovery of supergravity theories which live in higher than four space-time dimension, it has been a natural approach to reconsider the Kaluza-Klein idea in order to lower the number of spatial dimensions to the familiar three ones we are accostumed with. As presented in [8], we give here an outline of the general features of Kaluza-Klein theories and how the mass spectrum of a K-K theory can be found:

1. we consider our complete theory in d dimensions: we need to know the metric g_{MN} and all other fields.

2. We are interested in ground-state solutions of the quations of motion that show a spontaneous compactification to M_{d-k} , where M_{d-k} is a maximally symmetric space-time and M_k will be the compactified internal space. Which means we can see the d dimensional space-time M_d as the cross product of M_{d-k} with M_k . In other words, we want the ground-state metric $\dot{g}_{MN}$ to be decomposable into the direct product (or "warped-product" if there is a factor that multiplies one of the components of the metric) of the ground-state metrics of M_{d-k} ($\dot{g}_{\mu\nu}$) and M_k ($\dot{g}_{mn}$).

$$\dot{g}_{MN} = \begin{pmatrix} f\left(y\right)\dot{g}_{\mu\nu}\left(x\right) & 0 \\ 0 & \dot{g}_{mn}\left(y\right) \end{pmatrix} \tag{4.2}$$

   where x are space-time coordinates (coordinates of M_{d-k}), y are internal coordinates (coordinates of M_k) and $f\left(y\right)$ is the warping factor.

3. To find the mass spectrum in M_{d-k} of the theory we expand the fields to linear order in their perturbations around their background value. For example, for the

metric we write: $g_{MN} = \dot{g}_{MN} + h_{MN}$;
where h_{MN} is the fluctuation of the metric from its ground-state value.

4. Then we linearize the equations of motion by substituting the expansions of the fields and retaining only terms up to linear order in the perturbations.

5. Eventually, we expand the perturbations into harmonics on M_k which will give us the mass spectrum.

5 Type IIA supergravity on $AdS_7 \times M_3$

In this section, we are going to see in some detail how the solutions to the equations of motion of supergravity on a specific background can be found. Our method will make use of generalized complex geometry to translate the supersymmetry transformations into constraints on the geometry of space-time and to find their solutions. The reason why we care about the supersymmetry transformations is that there is a close relationship between them and the field equations of the theory. For example, it must be true that the supersymmetry variation of an equation of motion gives another expression which still satisfies the equations of motion. Actually, all we would need to find all the equations of motion is just one of those equations and the supersymmetry transformations. Another way the equations of motion can be deduced is by computing the commutators of local supersymmetry transformations of the fields. In fact, for the supersymmetry algebra to be closed we need the commutators to be a combination of local symmetries (general coordinate invariance, local supersymmetry and local gauge symmetry) and we need the equations of motion must be satisfied. So, if we construct a consistent (closed) supersymmetry algebra using the supersymmetry transformations of the fields, at the same time we will also obtain the equations of motion.

5.1 Geometrical aspects of supergravity

In supergravity theories, the presence of certain objects influences the geometrical structure of the background space where the theory lives. More concretely, the spinors one introduces when writing supersymmetry transformations imply topological conditions on the manifolds. The supersymmetry transformations themselves further constrain the plethora of compatible manifold by adding further requirements through differential conditions.

For example, the presence of a nowhere vanishing spinor restricts the structure group of the manifold, which is the group of the transition functions between charts on the manifold, because only transition functions which connects the charts in a specific way are allowed.

Since the supersymmetry transformations are hard to solve one can try to replace spinors with differential forms (requiring that they give the same reduction of the structure group) and rewrite SUSY transformations as differential equations on forms, which are usually easier to deal with.

5.1.1 Type II supergravity and complex geometry

Here we consider the case of type II supergravity. If we are interested in supersymmetric bosonic configurations, we can set the fermionic fields to zero; then the generic supersymmetry transformations are proportional to the bosonic fields and the two spinorial parameters.

Furthermore, we require that our solutions are $d-k$ vacua (states with no particles), which means that they are maximally symmetric. Maximal symmetry makes the metric block-diagonal and puts strong constraints on the allowed fields of the theory. The most general expression we can write down for such a metric would be:

$$ds^2_{10} = e^{2A(y)} ds^2_{M_{10-k}}(x) + ds^2_{M_k}(y) \ ; \tag{5.1}$$

where M_{10-k} is the external space with coordinates x and M_k is the internal space with coordinates y, and $A(y)$ is the warping factor. Also, we can split the gamma matrices Γ_M in a $10-k$ part Γ_μ and a k part Γ_m so that they act on tensor product space $M_{10-k} \oplus M_k$:

$$\Gamma_\mu = e^A \gamma_\mu \otimes 1 \, ; \Gamma_m = \gamma_5 \otimes \gamma_m \ . \tag{5.2}$$

The same splitting can be done with the spinorial parameter of the supersymmetry transformations which can be written as:

$$\epsilon = \sum_{i,I} \alpha_{iI} \zeta_i \otimes \eta_I \ , \tag{5.3}$$

where $i = 1, 4$ and $I = 1, 8$.

Again, the requirement of maximal symmetry constrains the external part ζ_i of the spinor to take certain values, so from the SUSY transformations we obtain a set of differential equations for the internal spinors η_I. However, the system of equations written in terms of spinorial quantities is usually very hard to solve anyway and so we turn to a different approach.

We can recongnize what kind of reduction of the structure group of the manifold is given by the spinors we have in the theory and find the differential forms that give the same reduction. For example, a spinor η on a six-dimensional internal manifold M_6 gives a reduction of the structure group to SU(3) (because SU(3)matrices leave the spinor invariant) and this reduction can also be obtained by defining two differential forms on the manifold, namely a real two-form J and a complex, decomposable, non-degenerate three-form Ω, which satisfy a compatibility condition. It can be proved that one is able to build invariant spinors with those forms and vice versa, showing that the two approaches are indeed equivalent. Once we know how to switch from spinors to forms we can rewrite the supersymmetry transformations as a set of differential equations on forms, which turns out to be more manageable and easier to work with.

In the case we have been dealing with, where we consider type II supergravity compactified to a four dimensional maximally symmetric space-time with the two internal spinors coinciding, the supersymmetry transformations, in the absence of fluxes, translate to differential conditions on J and Ω which make our manifold both complex and symplectic at the same time, or in other words they make it a be a Calabi-Yau manifold.

If we were to tackle the case where the two spinors are not equivalent, then we would face much greater difficulties. In this scenario, the structure group of the manifold is a function of the two spinors taken together, so at each point we could have different reductions depending on the relation between the spinors at that point (at some point on the manifold they may be equivalent, giving an SU(3) structure, while at other points they may not, giving SU(2) structure). When the structure group becomes dependent on the point of the manifold the translation from spinors to forms becomes excidingly difficult and a different approach is needed to make progress.

5.1.2 SU(3) structure

We give here an outline of the case we have talked about, showing the links between the two approaches.

Tensors

As we have said, we can define an SU(3) structure by using tensors. We start by defining an Almost Complex Structure (ACS): an ACS $I : TM \to TM$, where TM is the tangent bundle, is a tensor I_n^m such that:

- $I^2 = -1$;
- I_n^m is Hermitian (or $J \equiv gI$ is antisymmetric) .

What happens now is that a generic one-form ω can either be in the i-eigenbundle L ($I\omega = i\omega$) or in the $-i$-eigenbundle $\bar{L}$ ($I\omega = -i\omega$). In this way, I defines a U(3) structure that can be augmented to an SU(3) structure by defining a three form Ω with specific properties.

An ACS is then said to be integrable if the bundle L satisfies:

$$[L, L]_{Lie} \subset L \ . \tag{5.4}$$

I is then called a Complex Structure (CS). An alternative definition of integrable structures makes use of the form Ω we have mentioned. In particular, an integrable I is given by a precise Ω which has the following property: it exists a form W_5 such that

$$d\Omega = W_5 \wedge \Omega \ . \tag{5.5}$$

So, an SU(3) structure is defined by a pair (J, Ω) where J is a real two-form and Ω is a complex, non-degenrate, decomposable three-form such that: $J \wedge \Omega = 0$ and $J^3 = \frac{3}{4} i\Omega \wedge \bar{\Omega}$

Spinors

As we have seen, we can define an SU(3) structure by using a spinor on M_6: a six-dimensional spinor makes the manifold M_6 a spin-manifold ($M_6 = Spin\,(6) \cong SU\,(4)$); if the spinor is nowhere vanishing on M_6 we then need to consider only transition functions that preserve such SU(4) spinor: $\eta_\alpha = g_{\alpha\beta}\eta_\beta$ where η is the spinor and g is the transition function, has to hold. The identity is true for η SU(4) spinor only if $g \in SU\,(3)$.

5.1.3 An introduction to generalized complex geometry

Generalized complex geometry (GCG) is the tool we need to solve the problems we face when rephrasing SUSY transformations of spinorial objects to differential conditions on forms. With GCG, we switch focus from the structure group on the tangent bundle of the manifold to the structure group on the generalized tangent bundle (direct sum of tangent and cotangent bundle: $T \oplus T^*$). In this new space, the structure group is independent from the points of the manifold and so we can give a unique translation of the spinorial system in terms of new objects typical of this approach. In fact, when we consider the two spinors together to form bi-spinors, called pure spinors, we get a reduction of the structure group on the generalized tangent bundle to a $SU(3) \times SU(3)$. The same reduction can also be obtained through two generalized complex structures (GCS) $\mathcal{J}$. We will see how to relate pure spinors and GCS so that the requirement that two GCS define a $SU(3) \times SU(3)$ can be translated into pure spinor language. Eventually, one can rewrite the supersymmetry transformations using pure spinors to get a very simple system of equations.

5.1.4 $SU(3) \times SU(3)$ structure

Let's see what happens when we use generalized complex geometry in a little more details. Here we explore the links between the two different descriptions of an $SU(3) \times SU(3)$ structure. Remember that we are still considering the case of a six dimensional internal manifold M_6.

Tensors

An $SU(3) \times SU(3)$ structure can be defined by using tensors. We start by defining a Generalized Almost Complex Structure (GACS): an ACS $\mathcal{J} : T \oplus T^* \to T \oplus T^*$ is a map such that:

- $\mathcal{J}^2 = -1_{6\times 6}$;
- $\mathcal{J}$ is Hermitian or $\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$, with $\mathcal{I} = \begin{pmatrix} 0 & 1_6 \\ 1_6 & 0 \end{pmatrix}$.

Now, the elements of generalized tangent bundle, called generalized complex vectors X (which are a combination of a form and a vector field) can either belong to the i-eigenbundle $L_{\mathcal{J}}$ ($\mathcal{J}X = iX$) or in the $-i$-eigenbundle $\bar{L}_{\mathcal{J}}$ ($\mathcal{J}X = -iX$). In this way, $\mathcal{J}$ defines a U(3,3) structure on the generalized tangent bundle $T \oplus T^*$.

A GACS is then said to be integrable if the i-eigenbundle $L_{\mathcal{J}}$ satisfies:

$$[L_{\mathcal{J}}, L_{\mathcal{J}}]_{Courant} \subset L_{\mathcal{J}} \ ; \tag{5.6}$$

$\mathcal{J}$ is then called a Generalized Complex Structure (GCS). An alternative definition of integrable structures which links them to polyforms (pure spinors) is the following: $\mathcal{J}_\Phi$ is integrable if the associated pure spinor Φ is related to a generic form W on $T \oplus T^*$ through the identity:

$$d\Phi = W \cdot \Phi \ . \tag{5.7}$$

An $U(3) \times U(3)$ structure (which can then easily be reduced to a $SU(3) \times SU(3)$ structure) is defined by two commuting GACS ($[\mathcal{J}_1, \mathcal{J}_2] = 0$) which are compatible (they define a positive-definite metric through their product: $G \equiv -\mathcal{J}_1\mathcal{J}_2$).

Pure spinors

If we look at the differential operators on M_6 :

$$\Gamma = \left\{\partial_1 \llcorner, \ldots, \partial_6 \llcorner, dx^1\wedge, \ldots, dx^6\wedge\right\} \ ,$$

they form a Clifford algebra $Cliff(6,6)$ with respect to the metric $\mathcal{I} = \begin{pmatrix} 0 & 1_6 \\ 1_6 & 0 \end{pmatrix}$. Then, all differential forms can be seen as $Cliff(6,6)$ spinors. We call $\Phi_\pm$ differential forms of even/odd degree respectively.

A pure spinor Φ is a polyform whose annihilator space L_Φ (the space of all X of $T \oplus T^*$ such that $X \cdot \Phi = 0$) has maximal dimension (six in the case of M_6) and whose norm is non-zero. The general form of a pure spinor is:

$$\Phi = \Omega_k \wedge e^{B+iJ} \ ; \tag{5.8}$$

where Ω_k is a complex k-form and B, J are real two-form. We can now make the association $\mathcal{J} \to \Phi$, $L_\mathcal{J} \to L_\Phi$ to make sense of the integrability condition in terms of Φ previously written.

A pair of pure spinors is said to be compatible if the corresponding GACS are compatible and if they have the same norm. We also have the general result that every compatible pair of pure spinors $\Phi_\pm$ can be written as:

$$\Phi_\pm = e^{B\wedge}\eta^1_+ \otimes \eta^{2\dagger}_\pm \ ; \tag{5.9}$$

where B is a real two-form and $\eta^{1,2}$ are $Cl(6)$ spinors. In fact, the algebra $Cl(6,6)$ is found to be isomorphic to two copies of the algebra $Cl(6)$.

If such compatible pair of pure spinors exists, it reduces the structure group of $T \oplus T^*$ to $SU(3) \times SU(3)$.

5.2 $AdS_7 \times M_3$ solutions of type IIA supergravity

We have sketched how the supersymmetry transformations, which are intimately connected to the equations of motions, can be rewritten in a more simple way and then solved. We have mostly been concerned with the case of type II supergravity where the base-space takes the form $M_4 \times M_6$ and we now turn to the case where the background space is taken to be $AdS_7 \times M_3$.

The analysis has been done in [21], [22], [23]; in those works generalized complex geometry is used to rewrite the supersymmetry transformations for the case of II supergravity. It turns out that the system completely determines the form of the metridc and the fluxes. We list them here.

The metric is:

$$ds^2_{10} = e^{2A} ds^2_{AdS_7}(x) + ds^2_{M_3}(y) \ , \tag{5.10}$$

where A is the warping factor. It was also found that the internal space M_3 is an S^2-fibration over an interval. The zero form flux F_0 is the Romans mass and it is a constant. The three form flux H is:

$$H = -\left(6e^{-A} + F_0 x e^{\phi}\right) vol_{M_3} \ , \tag{5.11}$$

where x is a function related to the volume of the S^2 contained in M_3.

The two form flux F_2 is:

$$F_2 = \frac{1}{16} e^{A-\phi} \sqrt{1-x^2} \left(F_0 e^{A+\phi} x - 4\right) vol_{S^2} \ , \tag{5.12}$$

the four form flux F_4 is zero because otherwise it would break maximal symmetry in seven dimensions. In our treatment we will use these values of the metric and fluxes as background values and we will expand the corresponding fields in perturbations around them in order to linearize the equations of motion.

6 Linearized IIA supergravity equations of motion on $AdS_7 \times M_3$ backgorund

We will now turn to the task of linearizing the equations of motion of type IIA supergravity on the specific background we have considered. We are going to start from the known equations of motion of type IIA supergravity then we will expand the fields to first order in the fluctuations and keep only first order terms in the equations. Finally, we will evaluate the expressions found using the background value of the fields and metric for the case of $AdS_7 \times M_3$ background.

6.1 Bianchi identities

Let's start by looking at the Bianchi identities and their dual counterpart. As we have seen before, we can write the generic Bianchi identity in form language as:

$$(d + H\wedge)\, F = 0\ ; \tag{6.1}$$

more explicitly, for an R-R field strength F_p which is a p-form, the Bianchi identity is:

$$dF_p + H \wedge F_{p-2} = 0\ . \tag{6.2}$$

We will denote the background fields with a dot $\dot{}$ over their symbol. So, if we consider only the background fields the previous equation reads:

$$d\dot{F}_p + \dot{H} \wedge \dot{F}_{p-2} = 0\ ; \tag{6.3}$$

we also require:

$$d\dot{H} = 0\ . \tag{6.4}$$

Now, if we expand the fields to first order:

$$\begin{cases} F = \dot{F} + f \\ H = \dot{H} + db \end{cases} , \tag{6.5}$$

we can linearize the Bianchi identity:

$$d\dot{F}_p + \dot{H} \wedge \dot{F}_{p-2} + df_p + \dot{H} \wedge f_{p-2} + db \wedge \dot{F}_{p-2} = 0 \ . \tag{6.6}$$

Using the identity for the background fields found in equation (6.3) we get the most general expression for the linearized Bianchi identity:

$$df_p + \dot{H} \wedge f_{p-2} + db \wedge \dot{F}_{p-2} = 0 \ . \tag{6.7}$$

6.1.1 2-Form field strength

We can start our analysis by considering the case where F is the 2-form. The linearized equation becomes:

$$df_2 + db\dot{F}_0 = 0 \ , \tag{6.8}$$

where $\dot{H} \wedge f_0 = 0$.

The identity is satisfied for:

$$f_2 = dc_1 - b\dot{F}_0 \ . \tag{6.9}$$

If we consider the background fields only, we have:

$$d\dot{F}_2 + \dot{H} \wedge \dot{F}_0 = 0 \ , \tag{6.10}$$

which gives us the useful identity:

$$d\dot{F}_2 = -\dot{H} \wedge \dot{F}_0 \ . \tag{6.11}$$

6.1.2 4-Form field strength

Let's turn our attention to the next higher form, the 4-form F_4. In this case the linearized Bianchi identity is:

$$df_4 + \dot{H} \wedge f_2 + db \wedge \dot{F}_2 \ , \tag{6.12}$$

which holds for:

$$f_4 = dc_3 - b \wedge \dot{F}_2 + \dot{H} \wedge c_1 \ . \tag{6.13}$$

For the background fields only the Bianchi identity reads:

$$d\dot{F}_4 + \dot{H} \wedge \dot{F}_2 = 0 \; , \tag{6.14}$$

remembering that $\dot{F}_4 = 0$, this tells us:

$$\dot{H} \wedge \dot{F}_2 = 0 \; . \tag{6.15}$$

6.1.3 6-Form field strength

The linearized Bianchi identity in the case of the 6-form F_6 is:

$$df_6 + \dot{H} \wedge f_4 = 0 \; . \tag{6.16}$$

We can rewrite the equation knowing from (3.36) that $f_6 = - * f_4$:

$$- d * f_4 + \dot{H} \wedge f_4 = 0 \; . \tag{6.17}$$

Taking the Hodge dual of this equation we get:

$$- *d * f_4 + * \left(\dot{H} \wedge f_4 \right) = 0 \; . \tag{6.18}$$

Substituting the expression of f_4 found in (6.13) and defining: $d^\dagger \equiv *d* = \iota^M \nabla_M = g^{MN} \iota_M \nabla_N$ we get:

$$- d^\dagger dc_3 + d^\dagger \left(b \wedge \dot{F}_2 \right) - d^\dagger \left(\dot{H} \wedge c_1 \right) + * \left[\dot{H} \wedge \left(dc_3 - b \wedge \dot{F}_2 + \dot{H} \wedge c_1 \right) \right] = 0 \; , \tag{6.19}$$

$$- d^\dagger dc_3 + d^\dagger \left(b \wedge \dot{F}_2 \right) - d^\dagger \left(\dot{H} \wedge c_1 \right) + * \left(\dot{H} \wedge dc_3 \right) = 0 \; , \tag{6.20}$$

where we have used $\dot{H} \wedge \dot{H} = 0$ and $\dot{H} \wedge \dot{F}_2 = 0$.

Now, we can switch to index notation in order to be able to analyse the various cases where the free indices take value in the internal (M_3) or external (AdS_7) subspace. Let's see term by term what is the "translation" of this equation:

$$d^\dagger dc_3 = \frac{1}{3!} \nabla^M \nabla_{[N} c_{PQR]} \iota_M \left(dx^N \wedge dx^P \wedge dx^Q \wedge dx^R \right) = \frac{4}{3!} \nabla^M \nabla_{[M} c_{PQR]} \left(dx^P \wedge dx^Q \wedge dx^R \right) =$$

$$= \frac{4}{3!}\left[\frac{1}{4}\left(\nabla^M \nabla_M c_{[PQR]} - 3\nabla^M \nabla_{[P|} c_{M|QR]}\right)\right] dx^P \wedge dx^Q \wedge dx^R \ , \tag{6.21}$$

$$d^\dagger \left(b \wedge \dot{F}_2\right) = \frac{4!}{2!2!}\nabla^M b_{[NP}\dot{F}_{QR]}\iota_M \left(dx^N \wedge dx^P \wedge dx^Q \wedge dx^R\right) = 6{\cdot}4\nabla^M b_{[MP}\dot{F}_{QR]}\left(dx^P \wedge dx^Q \wedge dx^R\right) =$$

$$= 24\left(\frac{1}{2}\nabla^M b_{M[P}\dot{F}_{QR]} + \frac{1}{2}\nabla^M \dot{F}_{M[R}b_{PQ]}\right) dx^P \wedge dx^Q \wedge dx^R \ , \tag{6.22}$$

$$d^\dagger \left(\dot{H} \wedge c_1\right) = \frac{4!}{3!}\nabla^M \dot{H}_{[NPQ}c_{R]}\iota_M \left(dx^N \wedge dx^P \wedge dx^Q \wedge dx^R\right) = 4{\cdot}4\nabla^M \dot{H}_{[MPQ}c_{R]}\left(dx^P \wedge dx^Q \wedge dx^R\right) =$$

$$16\left[\frac{1}{4}\left(3\nabla^M \dot{H}_{M[PQ}c_{R]} - \nabla^M c_M \dot{H}_{[PQR]}\right)\right] dx^P \wedge dx^Q \wedge dx^R \ , \tag{6.23}$$

$$* \left(\dot{H} \wedge dc_3\right) = * \left(\frac{7!}{4!3!}\dot{H}_{[MNT}\nabla_U c_{VZS]} dx^M \wedge dx^N \wedge dx^T \wedge dx^U \wedge dx^V \wedge dx^Z \wedge dx^S\right) =$$

$$= 35\frac{\sqrt{-g}}{3!}\epsilon_{PQR}{}^{MNTUVZS}\dot{H}_{[MNT}\nabla_U c_{VZS]} dx^P \wedge dx^Q \wedge dx^R \ . \tag{6.24}$$

Where we have used the following identities:

$$\iota_M \left(dx^{M_1} \wedge ... \wedge dx^{M_k}\right) = k\delta_M^{[M_1} dx^{M_2} \wedge ... \wedge dx^{M_k]} \ , \tag{6.25}$$

$$A_{M_1...M_k} \wedge B_{N_1...N_j} = \frac{(k+j)!}{k!j!} A_{[M_1...M_K} B_{N_1...N_J]} \ , \tag{6.26}$$

$$* \left(A_{M_1 \dots M_k} dx^{M_1} \wedge \dots \wedge dx^{M_k} \right) = \frac{\sqrt{-g}}{(d-k)!} \epsilon_{M_{k+1} \dots M_d}{}^{M_1 \dots M_k} A_{M_1 \dots M_k} dx^{M_{k+1}} \wedge \dots \wedge dx^{M_d} \ . \quad (6.27)$$

d is the number of space-time dimensions.

Putting everything together, we can rewrite the linearized Bianchi identity in index notation:

$$-\frac{1}{3!} \left(\nabla^M \nabla_M c_{[PQR]} - 3 \nabla^M \nabla_{[P|} c_{M|QR]} \right) + 12 \left(\nabla^M b_{M[P} \dot{F}_{QR]} + \nabla^M \dot{F}_{M[R} b_{PQ]} \right) +$$

$$-4 \left(3 \nabla^M \dot{H}_{M[PQ} c_{R]} - \nabla^M c_M \dot{H}_{[PQR]} \right) + 35 \frac{\sqrt{-g}}{3!} \epsilon_{PQR}{}^{MNTUVZS} \dot{H}_{[MNT} \nabla_U c_{VZS]} = 0 \ . \quad (6.28)$$

We want to simplify the equation by imposing the gauge choice: $\nabla^m c_{mNP} = \nabla^m c_m = \nabla^m b_{mN} = 0$ (where small latin letters refer to internal indices). In order to do so, we have to exchange the covariant derivatives in the second term of this equation although this will introduce terms proportional to the Riemann tensor. Here is how the mentioned term can be rewritten:

$$\bar{g}^{MS} \nabla_S \nabla_{[P|} c_{M|QR]} = \bar{g}^{MS} \left(\nabla_{[P} \nabla_S c_{|M|QR]} + \left[\nabla_S, \nabla_{[P} \right] c_{|M|QR]} \right) =$$

$$= \bar{g}^{MS} \left(\nabla_{[P} \nabla_S c_{|M|QR]} - \bar{g}^{DF} \left(\bar{R}_{FMS[P|} c_{D|QR]} + \bar{R}_{F[Q|S|P|} c_{MD|R]} + \bar{R}_{F[R|S|P} c_{|M|Q]D} \right) \right) =$$

$$= \bar{g}^{MS} \left(\nabla_{[P} \nabla_S c_{|M|QR]} - \bar{g}^{DF} \left(\bar{R}_{FMS[P} c_{D|QR]} + 2 \bar{R}_{F[Q|S|P|} c_{MD|R]} \right) \right) =$$

$$= \nabla_{[P} \nabla^M c_{|M|QR]} - \left(-\bar{R}_{[P}{}^D c_{D|QR]} + 2 \bar{R}^D{}_{[Q}{}^M{}_{|P|} c_{MD|R]} \right) \ . \quad (6.29)$$

Rewriting $\bar{R}^D{}_{[Q}{}^M{}_{|P|}$ using the first Bianchi identity for the Riemann tensor: $R^M{}_{[NPQ]}$.

$$\bar{R}^D{}_{[Q}{}^M{}_P c_{R]MD} = \left(-\bar{R}^{DM}{}_{[PQ} - \bar{R}^D{}_{[PQ}{}^M \right) c_{R]MD} = ,$$

$$= \left(-\bar{R}^{DM}{}_{[PQ} - \bar{R}^{D}{}_{[Q}{}^{M}{}_{P}\right) c_{R]MD} \ . \tag{6.30}$$

Finally, we get:

$$2\bar{R}^{D}{}_{[Q}{}^{M}{}_{P}c_{R]MD} = -\bar{R}^{DM}{}_{[PQ}c_{R]MD} \ . \tag{6.31}$$

The commutator is then:

$$\bar{g}^{MS}\nabla_S\nabla_{[P|}c_{M|QR]} = \nabla_{[P}\nabla^{M}c_{|M|QR]} - \left(-\bar{R}_{[P}{}^{D}c_{D|QR]} + \bar{R}^{DM}{}_{[PQ}c_{R]DM}\right) \ . \tag{6.32}$$

We can now plug in equation (6.28) the expression we have just found and we get:

$$-\frac{1}{3!}\left[\nabla^{M}\nabla_{M}c_{[PQR]} - 3\nabla_{[P}\nabla^{M}c_{|M|QR]} + 3\left(-\bar{R}_{[P}{}^{D}c_{D|QR]} + \bar{R}^{DM}{}_{[PQ}c_{R]DM}\right)\right] + 12\left(\nabla^{M}b_{M[P}\dot{F}_{QR]} + \right.$$
$$\left. +\nabla^{M}\dot{F}_{M[R}b_{PQ]}\right) - 4\left(3\nabla^{M}\dot{H}_{M[PQ}c_{R]} - \nabla^{M}c_{M}\dot{H}_{[PQR]}\right) + 35\frac{\sqrt{-g}}{3!}\epsilon_{PQR}{}^{MNTUVZS}\dot{H}_{[MNT}\nabla_{U}c_{VZS]} = 0 \ . \tag{6.33}$$

The equation has 3 free indices that we can choose to be either in AdS_7 or M_3. Once we specify the various possibilities for the contracted indices we can also impose our gauge condition and find the final expression of the equation.

3 indices in AdS_7 , 0 indices in M_3

In this case all free indices belong to AdS_7 and the linearized Bianchi identity for the 6-form becomes:

$$-\frac{1}{3!}\left[\nabla^{M}\nabla_{M}c_{[\mu\nu\rho]} - 3\nabla_{[\mu}\nabla^{M}c_{|M|\nu\rho]} + 3\left(-\bar{R}_{[\mu}{}^{D}c_{D|\nu\rho]} + \bar{R}^{DM}{}_{[\mu\nu}c_{\rho]DM}\right)\right] + 12\left(\nabla^{M}b_{M[\mu}\dot{F}_{\nu\rho]} + \right.$$
$$\left. +\nabla^{M}\dot{F}_{M[\rho}b_{\mu\nu]}\right) - 4\left(3\nabla^{M}\dot{H}_{M[\mu\nu}c_{\rho]} - \nabla^{M}c_{M}\dot{H}_{[\mu\nu\rho]}\right) + 35\frac{\sqrt{-g}}{3!}\epsilon_{\mu\nu\rho}{}^{MNTUVZS}\dot{H}_{[MNT}\nabla_{U}c_{VZS]} = 0 \ . \tag{6.34}$$

We can rewrite the equation expliciting the metric tensor. In this way, we will be able to write the different terms that arise thanks to the form of the metric tensor. In fact, our metric is a warped-product type metric, which means that it is a cross product of the internal and external metric weighted by a warping factor. In our case the metric is:

$$\hat{g}_{MN}\left(x,y\right)=e^{2A(y)}g_{\mu\nu}\left(x\right)\oplus g_{mn}\left(y\right)\ . \tag{6.35}$$

Or equivalently, we can rescale the metric so that we separate terms with x-dependence (x is used for space-time coordinates) from terms with y-dependence (y is used for internal coordinates):

$$\bar{g}_{MN}=e^{-2A(y)}\hat{g}_{MN}=g_{\mu\nu}\left(x\right)\oplus e^{-2A(y)}g_{mn}\left(y\right)\ . \tag{6.36}$$

Equation (6.34) becomes:

$$\begin{aligned}
&-\frac{1}{3!}\bar{g}^{MS}\left(\nabla_S\nabla_M c_{[\mu\nu\rho]}-3\nabla_{[\mu}\nabla_S c_{|M|\nu\rho]}+3\bar{g}^{DF}\left(-\bar{R}_{M[\mu|SF}c_{D|\nu\rho]}+\bar{R}_{FS[\mu\nu}c_{\rho]DM}\right)\right)+\\
&+12\bar{g}^{MS}\left(\nabla_S b_{M[\mu}\dot{F}_{\nu\rho]}+\nabla_S\dot{F}_{M[\rho}b_{\mu\nu]}\right)-4\bar{g}^{MS}\left(3\nabla_S\dot{H}_{M[\mu\nu}c_{\rho]}-\nabla_S c_M\dot{H}_{[\mu\nu\rho]}\right)+\\
&+35\frac{\sqrt{-g}}{3!}\left(\bar{g}^{AM}\bar{g}^{BN}\bar{g}^{CT}\bar{g}^{DU}\bar{g}^{EV}\bar{g}^{FZ}\bar{g}^{GS}\right)\epsilon_{\mu\nu\rho ABCDEFG}\dot{H}_{[MNT}\nabla_U c_{VZS]}=0\ .
\end{aligned} \tag{6.37}$$

We will now specify internal and external indices. Latin letters $\{m,n,p,...\}$ will be used for internal indices while greek letters $\{\mu,\nu,\rho,...\}$ for external ones.

$$\begin{aligned}
&-\frac{1}{3!}g^{\tau\sigma}\left[\nabla_\sigma\nabla_\tau c_{[\mu\nu\rho]}-3\nabla_{[\mu|}\nabla_\sigma c_{|\tau|\nu\rho]}+3g^{\varphi\chi}\left(-\bar{R}_{\tau[\mu|\sigma\varphi}c_{\chi|\nu\rho]}+\bar{R}_{\chi\sigma[\mu\nu}c_{\rho]\varphi\tau}\right)\right]+\\
&-\frac{1}{3!}e^{2A(y)}g^{ms}\left(\nabla_s\nabla_m c_{[\mu\nu\rho]}-3\nabla_{[\mu}\nabla_s c_{|m|\nu\rho]}\right)+\frac{3!4!}{7!}35\frac{\sqrt{-g}}{3!}e^{-6A(y)}\epsilon_{\mu\nu\rho}{}^{mnt\sigma\tau\varphi\chi}\dot{H}_{[mnt]}\nabla_{[\sigma}c_{\tau\varphi\chi]}=0\ .
\end{aligned} \tag{6.38}$$

We can now use the gauge choice $\nabla^m c_{mNP}=\nabla^m c_m=\nabla^m b_{mN}=0$.

$$\begin{aligned}
&-\frac{1}{3!}\left[\nabla^\tau\nabla_\tau c_{[\mu\nu\rho]}-3\nabla_{[\mu|}\nabla^\tau c_{\tau|\nu\rho]}+3\left(-\bar{R}_{[\mu|}{}^{\chi}c_{\chi|\nu\rho]}+\bar{R}^{\chi\tau}{}_{[\mu\nu}c_{\rho]\chi\tau}\right)\right]+\\
&-\frac{1}{3!}e^{2A(y)}\nabla^m\nabla_m c_{[\mu\nu\rho]}+\frac{\sqrt{-g}}{3!}e^{6A(y)}\epsilon_{\mu\nu\rho}{}^{mnt\sigma\tau\varphi\chi}\dot{H}_{[mnt]}\nabla_{[\sigma}c_{\tau\varphi\chi]}=0\ .
\end{aligned} \tag{6.39}$$

2 indices in AdS_7 , 1 index in M_3

Specifying the possible values of the indices and neglecting the null terms from the equation (6.33) we get:

$$\nabla^\tau\nabla_\tau c_{[n\nu\rho]} + e^{2A(y)}\nabla^m\nabla_m c_{[n\nu\rho]} - 3\nabla_{[n}\nabla^\tau c_{|\tau|\nu\rho]} + \\ + 3e^{4A(y)}\left(-\bar{R}_{[n|}{}^m c_{m|\nu\rho]} + \bar{R}^{mq}{}_{[n\nu} c_{\rho]mq}\right) + 4e^{2A(y)}\nabla^m \dot{F}_{mn} b_{[\nu\rho]} = 0\ . \tag{6.40}$$

1 index in AdS_7 , 2 indices in M_3

Again, starting from (6.33) :

$$-\frac{1}{3!}\left[\nabla^M\nabla_M c_{[nq\rho]} - 3\nabla_{[n}\nabla^M c_{|M|q\rho]} + 3\left(-\bar{R}_{[n|}{}^D c_{D|q\rho]} + \bar{R}^{DM}{}_{[nq} c_{\rho]DM}\right)\right] +$$

$$+ 12\left(\frac{1}{3}\nabla^M b_{M\rho}\dot{F}_{nq} + \frac{2}{3}e^{2A(y)}\nabla^m \dot{F}_{m[n} b_{q]\rho}\right) - 4e^{2A(y)}\left(\frac{3}{3!}\nabla^m \dot{H}_{mnq} c_\rho\right) = 0 \tag{6.41}$$

.

Which becomes:

$$-\frac{1}{3!}\left[\nabla^\tau\nabla_\tau c_{[nq\rho]} + e^{2A(y)}\nabla^m\nabla_m c_{[nq\rho]} - 3\nabla_{[n}\nabla^\tau c_{|\tau|q\rho]} + 3e^{4A(y)}\left(-\bar{R}_{[n|}{}^m c_{m|q\rho]} + \bar{R}^{ms}{}_{[nq} c_{\rho]ms}\right)\right] +$$

$$+ 12\left(\frac{1}{3}\nabla^\tau b_{\tau\rho}\dot{F}_{nq} + \frac{2}{3}e^{2A(y)}\nabla^m \dot{F}_{m[n} b_{q]\rho}\right) - 4e^{2A(y)}\left(\frac{3}{3!}\nabla^m \dot{H}_{mnq} c_\rho\right) = 0\ . \tag{6.42}$$

0 indices in AdS_7 , 3 indices in M_3

At last, we consider equation (6.33) where all free indices are internal:

$$-\frac{1}{3!}\left[\nabla^M\nabla_M c_{[nqr]} - 3\nabla_{[n}\nabla^M c_{|M|qr]} + 3\left(-\bar{R}_{[n|}{}^D c_{D|qr]} + \bar{R}^{DM}{}_{[nq} c_{r]DM}\right)\right] +$$

$$+12\left(\nabla^{M}b_{M[n}\dot{F}_{qr]}+\nabla^{M}\dot{F}_{M[r}b_{nq]}\right)-4\left(3\nabla^{M}\dot{H}_{M[nq}c_{r]}-\nabla^{M}c_{M}\dot{H}_{[nqr]}\right)=0\ . \tag{6.43}$$

Writing out all possibilities we have:

$$-\frac{1}{3!}\left(\nabla^{\tau}\nabla_{\tau}c_{[nqr]}+e^{2A(y)}\nabla^{m}\nabla_{m}c_{[nqr]}-3\nabla_{[n}\nabla^{\tau}c_{|\tau|qr]}+3e^{4A(y)}\left(-\bar{R}_{[n|}{}^{d}c_{d|qr]}+\bar{R}^{dm}{}_{[qn}c_{r]dm}\right)\right)+$$

$$+12\left(\nabla^{\tau}b_{\tau[n}\dot{F}_{qr]}+e^{2A(y)}\nabla^{m}\dot{F}_{m[r}b_{nq]}\right)-4\left(3e^{2A(y)}\nabla^{m}\dot{H}_{m[nq}c_{r]}-\nabla^{\tau}c_{\tau}\dot{H}_{[nqr]}\right)=0\ . \tag{6.44}$$

6.1.4 8-Form field strength

In this section we are interested in finding the linearized Bianchi identities of the 8-form F_8. As in the previous case, we are going to consider the dual forms corresponding to F_6 and F_8, namely F_4 and F_2, and the dualized equation.

The linearized Bianchi is:

$$df_8+\dot{H}\wedge f_6=0\ . \tag{6.45}$$

Now we consider the dual forms: $f_8=*f_2$ and $f_6=-*f_4$.

The equation becomes:

$$d*f_2-\dot{H}\wedge *f_4=0\ . \tag{6.46}$$

Dualizing it we get:

$$d^{\dagger}f_2-6!4!\dot{H}\llcorner f_4=0\ , \tag{6.47}$$

where:

$$*\left(\dot{H}\wedge *f_4\right)=\epsilon_{ABCRSTUVZX}\dot{H}^{ABC}\epsilon^{MNPQRSTUVZ}f_{MNPQ}=$$

$$= 6!4!\delta_A^M \delta_B^N \delta_C^P \delta_X^Q \dot{H}^{ABC} f_{MNPQ} = 6!4!\dot{H}^{MNP} f_{MNPX} = 6!4!\dot{H} \llcorner f_4 \; .$$

Substituting the expressions for f_2 and f_4 into (6.47) we have:

$$d^\dagger dc_1 - d^\dagger \left(b\dot{F}_0\right) - 6!4!\dot{H}\llcorner\left(dc_3 - b \wedge \dot{F}_2 + \dot{H} \wedge c_1\right) = 0 \; . \tag{6.48}$$

At this point we would like to switch to index notation. Let's see how each term is written:

$$d^\dagger dc_1 = \nabla^M \nabla_{[M} c_{Q]} dx^Q = \frac{1}{2}\left(\nabla^M \nabla_M c_Q - \nabla^M \nabla_Q c_M\right) dx^Q \; , \tag{6.49}$$

$$d^\dagger \left(b\dot{F}_0\right) = 2\nabla^M b_{[MQ]} \dot{F}_0 dx^Q \; , \tag{6.50}$$

$$\dot{H}\llcorner dc_3 = \dot{H}^{MNP} \nabla_{[M} c_{NPQ]} dx^Q = \dot{H}^{MNP} \left(\frac{1}{4}\nabla_M c_{[NPQ]} - \frac{3}{4}\nabla_{[N} c_{|M|PQ]}\right) dx^Q \; , \tag{6.51}$$

$$\dot{H}\llcorner\left(b \wedge \dot{F}_2\right) = \frac{4!}{2!2!}\dot{H}^{MNP}\left(b_{[MN}\dot{F}_{PQ]}\right) dx^Q = 6\dot{H}^{MNP}\left(b_{[MN}\dot{F}_{PQ]}\right) dx^Q = 6\dot{H}^{MNP}\left(\frac{1}{2}b_{[MN}\dot{F}_{P]Q} + \frac{1}{2}b_{Q[M}\dot{F}_{NP]}\right) dx^Q \; , \tag{6.52}$$

$$\dot{H}\llcorner\left(\dot{H} \wedge c_1\right) = \frac{4!}{3!}\dot{H}^{MNP}\left(\dot{H}_{[MNP}c_{Q]}\right) dx^Q = 4\dot{H}^{MNP}\left(\frac{1}{4}\dot{H}_{[MNP]}c_Q - \frac{3}{4}\dot{H}_{Q[MN}c_{P]}\right) dx^Q \; . \tag{6.53}$$

Substituting the terms written in index notation in the linearized equation found in (6.48) we get:

$$\frac{1}{2}\left(\nabla^M \nabla_M c_Q - \nabla^M \nabla_Q c_M\right) - 2\nabla^M b_{[MQ]}\dot{F}_0 - 6!4!\dot{H}^{MNP}\left[\left(\frac{1}{4}\nabla_M c_{[NPQ]} - \frac{3}{4}\nabla_{[N} c_{|M|PQ]}\right) + \right.$$

$$-3\left(b_{[MN}\dot{F}_{P]Q}+b_{Q[M}\dot{F}_{NP]}\right)+\left(\dot{H}_{[MNP]}c_Q-3\dot{H}_{Q[MN}c_{P]}\right)\Big]=0\ . \tag{6.54}$$

We can rewrite the equation in such a way that we are able to use the gauge condition: $\nabla^m c_m = \nabla^m c_{mNP} = \nabla^m b_{mN} = 0 \quad .$

In order to do so we have to do the following computation:

$$\bar{g}^{MS}\nabla_S\nabla_Q c_M = \bar{g}^{MS}\nabla_Q\nabla_S c_M + \bar{g}^{MS}\left[\nabla_S,\nabla_Q\right]c_M =$$

$$=\bar{g}^{MS}\nabla_Q\nabla_S c_M + \bar{g}^{MS}R_{SQM}{}^{D}c_D = \bar{g}^{MS}\nabla_Q\nabla_S c_M + \bar{g}^{MS}\bar{g}^{DF}R_{SQMF}c_D\ . \tag{6.55}$$

Rewriting equation (6.54) with this modification we obtain:

$$\frac{1}{2}\left(\nabla^M\nabla_M c_Q - \nabla_Q\nabla^M c_M - R_Q{}^D c_D\right) - 2\nabla^M b_{[MQ]}\dot{F}_0 +$$
$$-6!4!\dot{H}^{MNP}\left[\left(\frac{1}{4}\nabla_M c_{[NPQ]} - \frac{3}{4}\nabla_{[N}c_{|M|PQ]}\right) - 3\left(b_{[MN}\dot{F}_{P]Q}+b_{Q[M}\dot{F}_{NP]}\right)+\left(\dot{H}_{[MNP]}c_Q-3\dot{H}_{Q[MN}c_{P]}\right)\right]=0\ . \tag{6.56}$$

This is the final expression for the linearized Bianchi identity, we only have to evaluate it for the various possible values of the indices. Here we only have one free index which can either be an internal or an external one. Let's see what comes out from these subcases.

1 index in Ads_7, 0 indices in M_3

We start from equation (6.56) with an external free index:

$$\frac{1}{2}\left(\nabla^M\nabla_M c_\mu - \nabla_\mu\nabla^M c_M - R_\mu{}^D c_D\right) - 2\nabla^M b_{[M\mu]}\dot{F}_0 +$$
$$-6!4!\dot{H}^{MNP}\left[\left(\frac{1}{4}\nabla_M c_{[NP\mu]} - \frac{3}{4}\nabla_{[N}c_{|M|P\mu]}\right) - 3b_{\mu[M}\dot{F}_{NP]} + \dot{H}_{[MNP]}c_\mu\right]=0\ . \tag{6.57}$$

Evaluating the contractions:

$$\frac{1}{2}\left(\nabla^\tau\nabla_\tau c_\mu + e^{-2A(y)}\nabla^m\nabla_m c_\mu - \nabla_\mu\nabla^\tau c_\tau - R_\mu{}^\tau c_\tau\right) - 2\nabla^\tau b_{[\tau\mu]}\dot{F}_0 +$$
$$- 6!4!e^{6A(y)}\dot{H}^{mnp}\left[\left(\frac{1}{4}\nabla_m c_{[np\mu]} - \frac{3}{4}\nabla_{[n}c_{|m|p\mu]}\right) - 3b_{\mu[m}\dot{F}_{np]} + \dot{H}_{[mnp]}c_\mu\right] = 0 \ . \tag{6.58}$$

If we now introduce the definition: $\dot{H}^{mnp} = h\epsilon^{mnp}$,the equation becomes:

$$\frac{1}{2}\left(\nabla^\tau\nabla_\tau c_\mu + e^{2A(y)}\nabla^m\nabla_m c_\mu - \nabla_\mu\nabla^\tau c_\tau - R_\mu{}^\tau c_\tau\right) - 2\nabla^\tau b_{[\tau\mu]}\dot{F}_0 +$$
$$- 6!4!e^{6A(y)}h\left[\epsilon^{mnp}\left(\frac{1}{4}\nabla_m c_{[np\mu]} - \frac{3}{4}\nabla_{[n}c_{|m|p\mu]}\right) - 3\epsilon^{mnp}b_{\mu[m}\dot{F}_{np]} + 3!hc_\mu\right] = 0 \ . \tag{6.59}$$

0 indices in Ads_7, 1 index in M_3

Now we consider equation (6.56) where the free index is an internal one:

$$\frac{1}{2}\left(\nabla^M\nabla_M c_q - \nabla_q\nabla^M c_M - R_q{}^D c_D\right) - 2\nabla^M b_{[Mq]}\dot{F}_0 +$$
$$-6!4!\dot{H}^{MNP}\left[\left(\frac{1}{4}\nabla_M c_{[NPq]} - \frac{3}{4}\nabla_{[N}c_{|M|Pq]}\right) - 3\left(b_{[MN}\dot{F}_{P]q} + b_{q[M}\dot{F}_{NP]}\right) + \left(\dot{H}_{[MNP]}c_q - 3\dot{H}_{q[MN}c_{P]}\right)\right] = 0 \ , \tag{6.60}$$

$$\frac{1}{2}\left(\nabla^\tau\nabla_\tau c_q + e^{2A(y)}\nabla^m\nabla_m c_q - \nabla_q\nabla^\tau c_\tau - e^{2A(y)}R_q{}^m c_m\right) - 2\nabla^\tau b_{[\tau q]}\dot{F}_0 +$$
$$-6!4!e^{6A(y)}\dot{H}^{mnp}\left[\left(\frac{1}{4}\nabla_m c_{[npq]} - \frac{3}{4}\nabla_{[n}c_{|m|pq]}\right) - 3\left(b_{[mn}\dot{F}_{p]q} + b_{q[m}\dot{F}_{np]}\right) + \left(\dot{H}_{[mnp]}c_q - 3\dot{H}_{q[mn}c_{p]}\right)\right] = 0 \ . \tag{6.61}$$

Using: $\dot{H}^{mnp} = h\epsilon^{mnp}$;

$$\frac{1}{2}\left(\nabla^\tau\nabla_\tau c_q + e^{2A(y)}\nabla^m\nabla_m c_q - \nabla_q\nabla^\tau c_\tau - e^{2A(y)}R_q{}^m c_m\right) - 2\nabla^\tau b_{[\tau q]}\dot{F}_0 +$$

$$-6!4!e^{6A(y)}h\left[\epsilon^{mnp}\left(\frac{1}{4}\nabla_m c_{[npq]}-\frac{3}{4}\nabla_{[n}c_{|m|pq]}\right)-3\epsilon^{mnp}\left(b_{[mn}\dot{F}_{p]q}+b_{q[m}\dot{F}_{np]}\right)+(3!hc_q-6hc_q)\right]=0\ , \tag{6.62}$$

$$\frac{1}{2}\left(\nabla^\tau\nabla_\tau c_q+e^{2A(y)}\nabla^m\nabla_m c_q-\nabla_q\nabla^\tau c_\tau-e^{2A(y)}R_q{}^m c_m\right)-2\nabla^\tau b_{[\tau q]}\dot{F}_0+$$
$$-6!4!e^{6A(y)}h\epsilon^{mnp}\left[\left(\frac{1}{4}\nabla_m c_{[npq]}-\frac{3}{4}\nabla_{[n}c_{|m|pq]}\right)-3\left(b_{[mn}\dot{F}_{p]q}+b_{q[m}\dot{F}_{np]}\right)\right]=0\ . \tag{6.63}$$

6.1.5 Summary of Bianchi identities

Here we present a summary of the linearized equations we have found from the Bianchi identities of the 6 and 8 form.

6-Form

3 AdS_7 **, 0** M_3

$$\nabla^\tau\nabla_\tau c_{[\mu\nu\rho]}-3\nabla_{[\mu|}\nabla^\tau c_{\tau|\nu\rho]}+3\left(-\bar{R}_{[\mu|}{}^\chi c_{\chi|\nu\rho]}+\bar{R}^{\chi\tau}{}_{[\mu\nu}c_{\rho]\chi\tau}\right)+$$
$$e^{2A(y)}\nabla^m\nabla_m c_{[\mu\nu\rho]}-\sqrt{-g}e^{6A(y)}\epsilon_{\mu\nu\rho}{}^{mnt\sigma\tau\varphi\chi}\dot{H}_{[mnt]}\nabla_{[\sigma}c_{\tau\varphi\chi]}=0\ . \tag{6.64}$$

2 AdS_7 **, 1** M_3

$$\nabla^\tau\nabla_\tau c_{[n\nu\rho]}+e^{2A(y)}\nabla^m\nabla_m c_{[n\nu\rho]}-3\nabla_{[n}\nabla^\tau c_{|\tau|\nu\rho]}+$$
$$+3e^{4A(y)}\left(-\bar{R}_{[n|}{}^m c_{m|\nu\rho]}+\bar{R}^{mq}{}_{[n\nu}c_{\rho]mq}\right)+4e^{2A(y)}\nabla^m\dot{F}_{mn}b_{[\nu\rho]}=0\ . \tag{6.65}$$

1 AdS_7 **, 2** M_3

$$\nabla^\tau\nabla_\tau c_{[nq\rho]} + e^{2A(y)}\nabla^m\nabla_m c_{[nq\rho]} - 3\nabla_{[n}\nabla^\tau c_{|\tau|q\rho]} + 3e^{4A(y)}\left(-\bar{R}_{[n|}{}^m c_{m|q\rho]} + \bar{R}^{ms}{}_{[nq}c_{\rho]ms}\right) +$$

$$- 24\left(\nabla^\tau b_{\tau\rho}\dot{F}_{nq} + 2e^{2A(y)}\nabla^m\dot{F}_{m[n}b_{q]\rho}\right) + 12e^{2A(y)}\nabla^m\dot{H}_{mnq}c_\rho = 0\ . \qquad (6.66)$$

0 AdS_7 **, 3** M_3

$$\nabla^\tau\nabla_\tau c_{[nqr]} + e^{2A(y)}\nabla^m\nabla_m c_{[nqr]} - 3\nabla_{[n}\nabla^\tau c_{|\tau|qr]} + 3e^{4A(y)}\left(-\bar{R}_{[n|}{}^d c_{d|qr]} + \bar{R}^{dm}{}_{[qn}c_{r]dm}\right) +$$

$$- 72\left(\nabla^\tau b_{\tau[n}\dot{F}_{qr]} + e^{2A(y)}\nabla^m\dot{F}_{m[r}b_{nq]}\right) + 72\left(3e^{2A(y)}\nabla^m\dot{H}_{m[nq}c_{r]} - \nabla^\tau c_\tau\dot{H}_{[nqr]}\right) = 0\ . \qquad (6.67)$$

8-Form

1 AdS_7 **, 0** M_3

$$\nabla^\tau\nabla_\tau c_\mu + e^{2A(y)}\nabla^m\nabla_m c_\mu - \nabla_\mu\nabla^\tau c_\tau - R_\mu{}^\tau c_\tau - 4\nabla^\tau b_{[\tau\mu]}\dot{F}_0 +$$

$$- 6!4!e^{6A(y)}h\left[\epsilon^{mnp}\left(\frac{1}{2}\nabla_m c_{[np\mu]} - \frac{3}{2}\nabla_{[n}c_{|m|p\mu]}\right) - 6\epsilon^{mnp}b_{\mu[m}\dot{F}_{np]} + 12hc_\mu\right] = 0\ . \qquad (6.68)$$

0 AdS_7 **, 1** M_3

$$\nabla^\tau \nabla_\tau c_q + e^{2A(y)} \nabla^m \nabla_m c_q - \nabla_q \nabla^\tau c_\tau - e^{2A(y)} R_q{}^m c_m - 4\nabla^\tau b_{[\tau q]} \dot{F}_0 + \\ - 6!4! e^{6A(y)} h \epsilon^{mnp} \left[\left(\frac{1}{2} \nabla_m c_{[npq]} - \frac{3}{2} \nabla_{[n} c_{|m|pq]} \right) - 6 \left(b_{[mn} \dot{F}_{p]q} + b_{q[m} \dot{F}_{np]} \right) \right] = 0 \; . \quad (6.69)$$

6.2 Dilaton equation

Now we turn to the task of linearizing equation (3.32) which describes the dilaton ϕ . We quote here the starting equation again:

$$\bar{R} + 4\nabla^2 \Phi - 4 (\nabla \Phi)^2 - \frac{1}{2} |H|^2 = 0 \; . \quad (6.70)$$

As before, we are going to expand the fields to linear order in the perturbations. They will take the following form

$$\bar{R} = \dot{\bar{R}} + \delta \bar{R} \; , \quad (6.71)$$

$$\Phi = \dot{\Phi} + \phi \; , \quad (6.72)$$

$$H = \dot{H} + db \; . \quad (6.73)$$

Where $\dot{R}$ and δR are obtained from the background value and the perturbation of the corresponding Riemann tensor.

The linearized equation for the dilaton Φ is then:

$$\dot{\bar{R}} + \delta \bar{R} + 4\nabla^2 \dot{\Phi} + 4\nabla^2 \phi - 4 \left(\nabla \dot{\Phi} \right)^2 - 4 (\nabla \phi)^2 - \frac{1}{2} |\dot{H}|^2 - |\dot{H} \cdot db| = 0 \; . \quad (6.74)$$

The equation of motion for the background fields is:

$$\dot{\bar{R}} + 4\nabla^2 \dot{\Phi} - 4 \left(\nabla \dot{\Phi} \right)^2 - \frac{1}{2} |\dot{H}|^2 = 0 \; . \quad (6.75)$$

We can use it to simplify the expression of (6.74), which becomes:

$$\delta\bar{R} + 4\nabla^2\phi - 4\left(\nabla\phi\right)^2 - |\dot{H}\cdot db| = 0\ . \tag{6.76}$$

Let's look explicitly to the expression of each term in index notation. We start by investigating the expression of the curvature scalar $\bar{R}$:

$$\delta\bar{R} = \delta\bar{R}_{MN}\bar{g}^{MN}\ , \tag{6.77}$$

$$\delta\bar{R}_{MN} = \frac{1}{2}\left(\hat{R}_{PM}h^P_N - \hat{R}^P{}_{NKM}h^K_P + \hat{R}^P{}_{MKN}h^K_P\right) - \frac{1}{2}\hat{\nabla}^K\hat{\nabla}_K h_{MN}\ , \tag{6.78}$$

$$\hat{R}_{PM} = \bar{R}_{PM} - (d-2)\left(\nabla_P\nabla_M A + \nabla_P A\nabla_M A - \hat{g}_{PM}\hat{g}^{RS}\nabla_R A\nabla_S A\right) - 2\hat{g}_{PM}\hat{g}^{RS}\nabla_R\nabla_S A\ , \tag{6.79}$$

$$\hat{R}^P{}_{NKM} = \bar{R}^P{}_{NKM} + 2\delta^P_{[N}\nabla_{K]}\nabla_M A - 2\hat{g}^{PR}\hat{g}_{M[N}\nabla_K\nabla_R A - 2\nabla_{[N}A\delta^P_{K]}\nabla_M A + 2\nabla_{[N}A\hat{g}_{K]M}\hat{g}^{PR}\nabla_R A + 2\hat{g}_{M[N}\delta^P_{K]}\hat{g}^{RS}\nabla_R A\nabla_S A\ . \tag{6.80}$$

So, putting everything together we get:

$$\begin{aligned}
\delta\bar{R}_{MN} = \frac{1}{2}\Big\{&\Big[\bar{R}_{PM} - (d-2)\left(\nabla_P\nabla_M A + \nabla_P A\nabla_M A - \hat{g}_{PM}\hat{g}^{RS}\nabla_R A\nabla_S A\right) - 2\hat{g}_{PM}\hat{g}^{RS}\nabla_R\nabla_S A\Big]h^P_N + \\
&- \Big[\bar{R}^P{}_{NKM} + 2\delta^P_{[N}\nabla_{K]}\nabla_M A - 2\hat{g}^{PR}\hat{g}_{M[N}\nabla_K\nabla_R A - 2\nabla_{[N}A\delta^P_{K]}\nabla_M A + 2\nabla_{[N}A\hat{g}_{K]M}\hat{g}^{PR}\nabla_R A + 2\hat{g}_{M[N}\delta^P_{K]}\hat{g}^{RS}\nabla_R A\nabla_S A\Big]h^K_P + \\
&+ \Big[\bar{R}^P{}_{MKN} + 2\delta^P_{[M}\nabla_{K]}\nabla_N A - 2\hat{g}^{PR}\hat{g}_{N[M}\nabla_K\nabla_R A - 2\nabla_{[M}A\delta^P_{K]}\nabla_N A + 2\nabla_{[M}A\hat{g}_{K]N}\hat{g}^{PR}\nabla_R A + 2\hat{g}_{N[M}\delta^P_{K]}\hat{g}^{RS}\nabla_R A\nabla_S A\Big]h^K_P\Big\} + \\
&- \frac{1}{2}\hat{\nabla}^K\hat{\nabla}_K h_{MN}\ .
\end{aligned} \tag{6.81}$$

The other terms, written in tensor notation, are:

$$\nabla^2\phi = \bar{g}^{MN}\nabla_M\nabla_N\phi \tag{6.82}$$

$$(\nabla\phi)^2 = \bar{g}^{MN}\nabla_M\phi\nabla_N\phi \tag{6.83}$$

$$\dot{H}\cdot db = \bar{g}^{MN}\bar{g}^{PQ}\bar{g}^{RS}\dot{H}^{[MPR]}2\nabla_{[N}b_{QS]} \tag{6.84}$$

Now, the linearized dilaton equation (6.76) becomes:

$$\begin{aligned}
&\Bigg\{\frac{1}{2}\Bigg\{\Big[\bar{R}_{PM} - (d-2)\Big(\nabla_P\nabla_M A + \nabla_P A\nabla_M A - \hat{g}_{PM}\hat{g}^{RS}\nabla_R A\nabla_S A\Big) - 2\hat{g}_{PM}\hat{g}^{RS}\nabla_R\nabla_S A\Big]\, h^P_N + \\
&- \Big[\bar{R}^P{}_{NKM} + 2\delta^P_{[N}\nabla_{K]}\nabla_M A - 2\hat{g}^{PR}\hat{g}_{M[N}\nabla_K\nabla_R A - 2\nabla_{[N}A\delta^P_{K]}\nabla_M A + 2\nabla_{[N}A\hat{g}_{K]M}\hat{g}^{PR}\nabla_R A + 2\hat{g}_{M[N}\delta^P_{K]}\hat{g}^{RS}\nabla_R A\nabla_S A\Big]\, h^K_P + \\
&+ \Big[\bar{R}^P{}_{NKM} + 2\delta^P_{[M}\nabla_{K]}\nabla_N A - 2\hat{g}^{PR}\hat{g}_{N[M}\nabla_K\nabla_R A - 2\nabla_{[M}A\delta^P_{K]}\nabla_N A + 2\nabla_{[M}A\hat{g}_{K]N}\hat{g}^{PR}\nabla_R A + 2\hat{g}_{N[M}\delta^P_{K]}\hat{g}^{RS}\nabla_R A\nabla_S A\Big]\, h^K_P\Bigg\} + \\
&-\frac{1}{2}\hat{\nabla}^K\hat{\nabla}_K h_{MN}\Bigg\}\bar{g}^{MN} + 4\bar{g}^{MN}\nabla_M\nabla_N\phi - 8\bar{g}^{MN}\nabla_M\phi\nabla_N\dot{\Phi} - 2\bar{g}^{MN}\bar{g}^{PQ}\bar{g}^{RS}\dot{H}^{[MPR]}\nabla_{[N}b_{QS]} = 0\,.
\end{aligned} \tag{6.85}$$

6.3 Einstein equation

In this subsection we look at the Einstein equation, we quote it here again:

$$e^{-2\Phi}\left(\bar{R}_{MN} + 2\nabla_M\nabla_N\Phi - \frac{1}{2}H_M^{\ PQ}H_{NPQ}\right) - \frac{1}{4}\sum_{p\geq 2}|F_p|^2_{MN} = 0\,, \tag{6.86}$$

where: $|F_p|^2_{MN} = \frac{1}{(p-1)!}F_M^{\ Q_1\ldots Q_{p-1}}F_{NQ_1\ldots Q_{p-1}}$.
To linearize the equation we expand the fields to linear order in the perturbations:

$$\bar{R}_{MN} = \dot{\bar{R}}_{MN} + \delta\bar{R}_{MN}\,, \tag{6.87}$$

$$\Phi = \dot{\Phi} + \phi\,, \tag{6.88}$$

$$H = \dot{H} + db\,, \tag{6.89}$$

$$F_p = \dot{F}_p + f_p \ . \tag{6.90}$$

The linearized equation is:

$$e^{-2\dot{\Phi}} (1 - 2\phi) \left[\dot{\bar{R}}_{MN} + \delta\bar{R}_{MN} + 2\nabla_M\nabla_N\dot{\Phi} + 2\nabla_M\nabla_N\phi - \frac{1}{2}\bar{g}^{PR}\bar{g}^{QS}\left(\dot{H}_{MRS}\dot{H}_{NPQ} + \nabla_{[M}b_{RS]}\dot{H}_{NPQ} + \dot{H}_{MRS}\nabla_{[N}b_{PQ]}\right)\right] +$$

$$- \frac{1}{4}\sum_{p\geq 2}\left(|\dot{F}_p|^2_{MN} + 2|\dot{F}_p \cdot f_p|_{MN}\right) = 0 \ . \tag{6.91}$$

For the background fields we have:

$$e^{-2\dot{\Phi}}\left[\dot{\bar{R}}_{MN} + 2\nabla_M\nabla_N\dot{\Phi} - \frac{1}{2}\bar{g}^{PR}\bar{g}^{QS}\left(\dot{H}_{MRS}\dot{H}_{NPQ}\right)\right] - \frac{1}{4}\sum_{p\geq 2}|\dot{F}_p|^2_{MN} = 0 \ . \tag{6.92}$$

The linearized Einstein equation in tensor notation, where we have used the identity for the background fields is:

$$e^{-2\dot{\Phi}}\left[\delta\bar{R}_{MN} + 2\nabla_M\nabla_N\phi - \frac{1}{2}\bar{g}^{PR}\bar{g}^{QS}\left(\nabla_{[M}b_{RS]}\dot{H}_{NPQ} + \dot{H}_{MRS}\nabla_{[N}b_{PQ]}\right)\right] +$$

$$-2\phi e^{-2\dot{\Phi}}\left[\dot{\bar{R}}_{MN} + 2\nabla_M\nabla_N\dot{\Phi} - \frac{1}{2}\bar{g}^{PR}\bar{g}^{QS}\left(\dot{H}_{MRS}\dot{H}_{NPQ}\right)\right] +$$

$$-\frac{1}{4}\left[\bar{g}^{QP}\left(\dot{F}_{MQ}\dot{F}_{NP}\right) + \bar{g}^{QP}\left(\dot{F}_{MQ}f_{NP}\right) + \bar{g}^{QP}\left(f_{MQ}\dot{F}_{NP}\right) + \frac{1}{3!}\bar{g}^{QP}\bar{g}^{RT}\bar{g}^{SU}\left(\dot{F}_{MQRS}\dot{F}_{NPTU}\right) +\right.$$

$$+\frac{1}{3!}\bar{g}^{QP}\bar{g}^{RT}\bar{g}^{SU}\left(\dot{F}_{MQRS}f_{NPTU}\right) + \frac{1}{3!}\bar{g}^{QP}\bar{g}^{RT}\bar{g}^{SU}\left(f_{MQRS}\dot{F}_{NPTU}\right) +$$

$$+\frac{1}{5!}\bar{g}^{QP}\bar{g}^{RT}\bar{g}^{SU}\bar{g}^{VX}\bar{g}^{ZY}\left(\dot{F}_{MQRSVZ}\dot{F}_{NPTUXY}\right) + \frac{1}{5!}\bar{g}^{QP}\bar{g}^{RT}\bar{g}^{SU}\bar{g}^{VX}\bar{g}^{ZY}\left(f_{MQRSVZ}\dot{F}_{NPTUXY}\right) +$$

$$+\frac{1}{5!}\bar{g}^{QP}\bar{g}^{RT}\bar{g}^{SU}\bar{g}^{VX}\bar{g}^{ZY}\left(\dot{F}_{MQRSVZ}f_{NPTUXY}\right) + \frac{1}{7!}\bar{g}^{QP}\bar{g}^{RT}\bar{g}^{SU}\bar{g}^{VX}\bar{g}^{ZY}\bar{g}^{AF}\bar{g}^{BG}\left(\dot{F}_{MQRSVZAB}\dot{F}_{NPTUXYFG}\right) +$$

$$+\frac{1}{7!}\bar{g}^{QP}\bar{g}^{RT}\bar{g}^{SU}\bar{g}^{VX}\bar{g}^{ZY}\bar{g}^{AF}\bar{g}^{BG}\left(\dot{F}_{MQRSVZAB}f_{NPTUXYFG}\right) + \frac{1}{7!}\bar{g}^{QP}\bar{g}^{RT}\bar{g}^{SU}\bar{g}^{VX}\bar{g}^{ZY}\bar{g}^{AF}\bar{g}^{BG}\left(f_{MQRSVZAB}\dot{F}_{NPTUXYFG}\right) +$$

$$+\frac{1}{9!}\bar{g}^{QP}\bar{g}^{RT}\bar{g}^{SU}\bar{g}^{VX}\bar{g}^{ZY}\bar{g}^{AF}\bar{g}^{BG}\bar{g}^{CH}\bar{g}^{DE}\left(\dot{F}_{MQRSVZABCD}\dot{F}_{NPTUXYFGHE}\right) +$$

$$+\frac{1}{9!}\bar{g}^{QP}\bar{g}^{RT}\bar{g}^{SU}\bar{g}^{VX}\bar{g}^{ZY}\bar{g}^{AF}\bar{g}^{BG}\bar{g}^{CH}\bar{g}^{DE}\left(\dot{F}_{MQRSVZABCD}f_{NPTUXYFGHE}\right) +$$

$$\left.+\frac{1}{9!}\bar{g}^{QP}\bar{g}^{RT}\bar{g}^{SU}\bar{g}^{VX}\bar{g}^{ZY}\bar{g}^{AF}\bar{g}^{BG}\bar{g}^{CH}\bar{g}^{DE}\left(f_{MQRSVZABCD}\dot{F}_{NPTUXYFGHE}\right)\right] = 0 \ . \tag{6.93}$$

7 Conclusions

We set out to linearize the equations of motion of type IIA supergravity on a $AdS_7 \times M^3$ background and we discovered most of them. We started from IIA supergravity equations of motion, we substituted the expressions of the fields expanded to linear order in the fluctuations around their background values and eventually we found the linearized equations of motion by keeping only terms up to first order. At the end, we succesfully computed the linearized Bianchi identities, summarized in 6.1.5, and we have put the basis for the computations of the dilaton and Einstein linearized equations in 6.2 and 6.3 respectively. The equations we have obtained have several traits in common with the ones found in [8] and [9], which describe cases similar to the one we have treated; this gives us strong hints that we are heading in the right direction.

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